STOCHASTIC SCENARIO-BASED TIME-DEPENDENT SHORTEST PATH

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Uncertainty is a fundamental property of transportation networks which evolve continually due to varying travel demand, traffic capacity, or individual behaviour. Over the past few years, the concept of strategic scenario based traffic assignment has emerged as a promising method to systematically incorporate these uncertainties into transport models (Hamdouch *et al.*, 2004; Marcotte *et al.*, 2004; Waller *et al.*, 2013; Dixit *et al.*, 2013). In strategic traffic assignment, the travel demand is modelled as a random variable which can take a finite number of values (scenarios), where each scenario corresponds to a representation of the network state. Hence in a strategic traffic assignment, users minimize their expected travel time while recognizing the underlying probability distribution for the scenarios and the network travel times in each scenario. A strategic dynamic traffic assignment problem requires that users be able to find the Shortest Path (SP) in a time-dependent network across a set of stochastic demand scenarios. Whilst efficient procedures exist to find optimal routes in deterministic and/or time-invariant environments (Dijkstra, 1959; Dreyfus, 1969; Frank, 1969), the same algorithms cannot always provide optimal solutions in stochastic, time-dependent networks efficiently. This work seeks to address this gap by finding the Stochastic Scenario-based Time-Dependent Shortest Path (SSTDSP) across a set of stochastic scenarios, where time-dependent arc costs are known in each scenario (Figure 1).

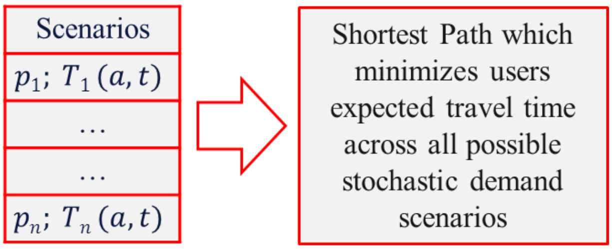

Figure 1: Scenario-based Stochastic Time-Dependent Shortest Path. p_i is the probability and $T_i(a,t)$ is the travel time on arc a at departure time t in the scenario i.

Due to the time-varying nature of transit systems and highways, the SP variant where arc costs represent travel times which depend on the time of arrival at the tail of the arc has attracted the attention of many researchers in the field of transportation science. Dreyfus (1969) was the first to observe that Dijkstra's algorithm (1959) could be used to solve the Time-Dependent Shortest Path (TDSP) with the same (polynomial time) complexity, given an optimal waiting policy at intermediate nodes: such waiting policy assumes that a traveller will always wait for the departure time which provides the earliest arrival time at the next node. However, the relaxations of this assumption has a severe impact on the complexity of the TDSP algorithms as observed by Orda and Rom (1990), which explored the role of delay (travel time) functions on arcs and different waiting policies. In particular, the TDSP is known to be a NP-hard problem if the network is not First-In First-Out (FIFO) and if the waiting policy is not optimal (Sherali *et al.* 1998). Furthermore, the case where delay functions are continuous may potentially result in a non-polynomial complexity of the TDSP problem in FIFO networks. This result was recently proven for piecewise linear delay functions by Foschini *et al.* (2011) who settled a conjecture established by Dean (2004).

If the network is assumed time-invariant, one may find the stochastic SP by simply determining the expected cost of each arc in the network and using a generic SP algorithm with the obtained arc costs (Frank, 1969). In this context, Waller and Ziliaskopoulos (2002) showed that polynomial time recourse algorithms can be obtained in the case where arc costs exhibit a limited spatial and/or temporal dependency. In contrast, the Stochastic Time Dependent Shortest Path (STDSP) was first studied by Hall (1986) who showed that this procedure cannot always find the least expected time path in stochastic, time-dependent networks. Psaraftis and Tsitsiklis (1993) studied a variant of the STDSP where the state of successive arcs are revealed upon arrival at a node and consider a routing

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policy where waiting time at nodes is permitted but penalized. They were able to obtain efficient algorithms given the assumption that the network is acyclic.

Miller-Hooks and Mahmassani (1998) investigated a more general case of this problem with no assumptions regarding the network FIFO property or waiting policies, and established an efficient algorithm to find the least possible time path in stochastic, time-dependent networks. However, by choosing the least arc cost at every node they ignored the impact of the conditionality of the state probabilities of travel times. Addressing this particular issue, Miller-Hooks and Mahmassani (2000) provided an optimal procedure to find the least expected time path, and introduced the notion of label dominance for stochastic, time-dependent networks. In order to ensure global optimality, the proposed algorithm must maintain an arbitrarily large but finite list of non-dominated labels at each node of the network. As the number of non-dominated labels at a node can grow exponentially (depending on the number of paths travelling through the node of interest), the aforementioned algorithm has a non-polynomial time complexity. Miller-Hooks (2001) also investigated the adaptive case, where arc cost information are revealed at the arrival at intermediate nodes and proposed an efficient algorithm for this variant. A thorough comparison of *a priori* and online routing policies in non-FIFO networks is provided in Miller-Hooks and Mahmassani (2003). A review of the online routing variants in stochastic, time-dependent networks is provided by Gao and Chabini (2006) who propose a two-dimensional taxonomy of STDSP problems according to arc cost dependency and information access. Their findings confirm that the assumptions in the modelling of stochastic, time-dependent networks can have a significant impact on the computational efficiency of routing algorithms.

This study investigates the efficiency of routing algorithms in stochastic, time-dependent networks with respect to different probability structures on arc costs as well as different node waiting policies. The main contribution of this work is to characterize the SSTDSP, which is a variant of the STDSP problem where time-dependent arc costs are organized by scenarios, and each scenario has an associated probability of occurrence. The SSTDSP can be seen as the least expected time-path across a finite set of stochastic scenarios, in which the link travel times are stochastically dependent. Namely, for a given departure time we assume that only one scenario can be realized. In this context, there is a complete stochastic dependency of the link travel times in contrast to the more general STDSP, where the random variables representing the link travel times are assumed to be independent.

Formally, let $G = \left(N, A, T, K, \left[c_{ij}^{tk}\right]\right)$ be a network where N is the set of nodes, A is the set of arcs, T is the set of time intervals and K is a vector of scenario probabilities such that $\sum_{k\in K} p_k = 1$ and $[c_{ij}^{tk}]$ is a four dimensional array representing the travel time on link (i, j) at time t in scenario k. Without any loss of generality, we seek the SSTDSP for a single origin-destination pair. Each stochastic scenario can be represented by a set of travel times $\tau_{ij}^{k}(t)$, for all link $(i, j) \in A$, arrival time $t \in T$ and scenario $k \in K$. Hence, for each scenario we are able to determine the travel time through a given path from any node to the destination node using the generic recursive formulation of time-dependent networks. Namely, let $\lambda_i^{\mu k}(t)$ be the label at node $i \in \Gamma^-(j)$, through path μ, in scenario k at time t, we have:

$$\lambda_i^{\mu k}(t) = \tau_{ij}^{k}(t) + \lambda_j^{\mu k}(t + \tau_{ij}^{k}(t))$$

Under the strategic assignment assumptions which require a complete stochastic dependency of the link travel times, the expected travel time to the destination through path μ, from node i at time t is:

$$E[\lambda_i^{\mu}(t)] = \sum_{k\in K} p_k \lambda_i^{\mu k}(t) = \sum_{k\in K} p_k \left(\tau_{ij}^{k}(t) + \lambda_j^{\mu k}\left(t + \tau_{ij}^{k}(t)\right)\right)$$

which can be also expressed as the sum of two expected values:

$$E[\lambda_i^{\mu}(t)] = E[\tau_{ij}(t)] + E[\lambda_j^{\mu}(t + \tau_{ij}(t))]$$

The first term, $E\left[\tau_{ij}(t)\right] = \sum_k p_k \tau_{ij}^k(t)$, gives the expected value of the travel time on link (i,j), whereas the second term, $E[\lambda_j^\mu(t + \tau_{ij}(t))]$, gives the expected value of the remaining subpath, from node j to the destination node, through path μ. In order to find the SSTDSP across a set of stochastic scenarios, we seek to determine the node labels such that the least expected time-subpath from every intermediary node is used to reach the destination node. We propose to determine the node labels using the following recursive equation:

$$\forall i \in \Gamma^-(j), \qquad \lambda_i(t) = E\left[\tau_{ij}(t)\right] + \min_{\mu}\left\{E\left[\lambda_j^\mu(t + \tau_{ij}(t))\right]\right\}$$

The optimal labels at node i can then be obtained by comparing each newly found vector label $[\lambda_i']_{t\in T}$ with the current vector label $[\lambda_i]_{t\in T}$ and keeping the non-dominated solutions (as defined in Miller-Hooks and Mahmassani, 2000). Therefore the node-labels are upper bound on the least expected time-path across all scenarios until the algorithm has examined all the nodes of the network. The algorithm used to find the SSTDSP can be executed backwards from the destination node, for which the optimal label is set to zero, and by examining each predecessor of the current node and updating the corresponding node-labels. Observe that only a bounded number of labels can be obtained at each node.

If waiting is not permitted at intermediary nodes, the above algorithm may require that of list of non-dominated labels be maintained at each node so as to keep track of the most efficient routing strategy. In this case, an upper bound on the least expected time-path can be obtained by decomposing the original network into $|K|$ deterministic time-dependent subnetworks $G_k = \left(N, A, T, \left[c_{ij}^{tk}\right]\right)$ where the stochastic scenario $k \in K$ is realized. Let T_k be the travel time of the TDSP in G_k for the origin-destination pair of interest. Let T_{UB} be defined as:

$$T_{UB} = \min_{\mathrm{k}} T_k$$

T_{UB} is an upper bound on the value of SSTDSP for the origin-destination pair of interest in G and can be used to prune the list of non-dominated node labels. A lower bound can be obtained using an adapted version of the Expected Lower Bound (ELB) algorithm introduced by Miller-Hooks and Mahmassani (2000). The ELB algorithm determines a lower bound on the a priori least expected time-path in polynomial time by ignoring the dominance among paths in the work. The lower bound on the least expected time-path can be used to measure the quality of the upper bound. Both bounds can also be used to derive efficient heuristic algorithms to find the least expected time-path under strategic assignment assumptions and measure the progress of these solution methods.

If waiting is permitted at intermediary nodes, the algorithm proposed can be adjusted to produce an efficient procedure able to solve the SSTDSP problem in polynomial time under specific behavioural assumptions. In particular, waiting time at an intermediary node can be introduced so as to obtain the least expected time-subpath from this node to the destination node. This strategic waiting policy implies that users are allowed to wait at an intermediary node on their path and are able to determine the optimal waiting time so as to minimize the expected on the remaining subpath to the destination. Let $\left[w_{ij}(t)\right]_{k\in K}$ be a vector of waiting times representing the waiting times at node j in each scenario k, when a user arrives at node $i \in \Gamma^-(j)$ at time t, we define:

$$\left[w_{ij}^*(t)\right]_{k\in K} = \arg \min_{\left[w_{ij}(t)\right]_{k\in K}} \left\{E\left[\lambda_j(t + w_{ij}(t) + \tau_{ij}(t))\right]\right\}$$

$\left[w_{ij}^*(t)\right]_{k\in K}$ is the optimal waiting policy in each scenario k at a given time t, such that the expected travel time on the subpath from j to the destination node is minimized. The node-labels can then be determined using the equation:

$$\forall i \in \Gamma^{-}(j), \qquad \lambda_i(t) = E\left[\tau_{ij}(t)\right] + E\left[\lambda_j(t + w_{ij}^{*}(t) + \tau_{ij}(t))\right]$$

The optimal labels at node i are then obtained by comparing each newly found vector label $[\lambda_i']_{t\in T}$ with the current vector label $[\lambda_i]_{t\in T}$ and keeping the minimum labels for each arrival time. Since only the minimum values are maintained, a single vector label is sufficient to find the SSTDSP under this waiting policy.

In this work, we evaluate the performance of the proposed algorithm for the SSTDSP when no waiting is permitted at intermediary nodes as well as when the strategic waiting policy is applied. Our results show that the algorithmic solutions obtained can be used to solve the aforementioned strategic dynamic traffic assignment problem in a computationally tractable manner and provide a novel approach to study strategic routing decisions in transportation networks.

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