Is equilibrium in transport pure Nash, mixed or Stochastic?

Vinayak V. Dixit [a],*, Laurent Denant-Boemont [b]

[a] Research Centre for Integrated Transport Innovation (rCITI), School of Civil and Environmental Engineering, University of New South Wales, Australia
[b] Department of Economics, CREM Université de Rennes 1, Rennes, France

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ABSTRACT

The classical theory of transport equilibrium is based on the Wardrop's first principle that describes a Nash User Equilibrium (UE), where in no driver can unilaterally change routes to improve his/her travel times. A growing number of economic laboratory experiments aiming at testing Nash-Wardrop equilibrium have shown that the Pure Strategy Nash Equilibrium (PSNE) is not able to explain the observed strategic choices well. In addition even though Mixed Strategy Nash Equilibrium (MSNE) has been found to fit better the observed aggregate choices, it does not explain the variance in choices well. This study analyses choices made by users in three different experiments involving strategic interactions in endogenous congestion to evaluate equilibrium prediction. We compare the predictions of the PSNE, MSNE and Stochastic User Equilibrium (SUE). In SUE, the observed variations in choices are assumed to be due to perception errors. The study proposes a method to iteratively estimate SUE models on choice data with strategic interactions. Among the three sets of experimental data the SUE approach was found to accurately predict the average choices, as well as the variances in choices. The fact that the SUE model was found to accurately predict variances in choices, suggests its applicability for transport equilibrium models that attempt to evaluate reliability in transportation systems. This finding is fundamental in the effort to determining a behaviourally consistent paradigm to model equilibrium in transport networks. The study also finds that Fechner error which is the inverse of the scale parameter in the SUE model is affected by the group sizes and the complexity of the cost function. In fact, the larger group sizes and complexity of cost functions increased the variability in choices. Finally, from an experimental design standpoint we show that it is not possible to estimate a noise parameter associate to Fechner error in the case when the choices are equally probable.

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1. Introduction

Transportation planning models have hinged on the classical theories of transport equilibrium that are based on Wardrop's principles (Wardrop, 1952), which is essentially a Nash equilibrium in which all users minimize their travel cost and have no incentives to change their chosen routes. Due to the extensive reliance on transportation planning models for allocation of billions of dollars (euros) of funds on transportation improvements, it has become critical to evaluate theories that underlie these models. Over the past several decades, other transportation equilibrium models such as the Stochastic User Equilibrium model (Daganzo and Sheffi, 1977) and Mixed Strategy Nash Equilibrium model (Bell, 2000) have been

* Corresponding author.
    E-mail addresses: v.dixit@unsw.edu.au (V.V. Dixit), boemont@univ-rennes1.fr (L. Denant-Boemont).

proposed. Stochastic User Equilibrium is a state of traffic flows where no user may lower their perceived travel time by uni-laterally changing their current route, where in the perceived travel time may be associated with random errors. In Mixed Strategy Nash Equilibrium, the user is unable to reduce their expected trip cost by changing their strategies, which are the user's path choice probabilities.

In recent year, several agencies have increased their focus on improving travel time reliability in networks. Variability of link travel time in a network can be attributed to variability in traffic demand (Hanson and Huff, 1988; Pas and Sundar, 1995; Stopher et al., 2008), capacity or supply (Gayah et al., 2013; Kim and Mahmassani, 2014), route choice behavior (Mirchandani and Soroush, 1987; Uchida and Iida, 1993; Jha et al., 1998; Lo and Tung, 2000; Bell, 2000; Yin and Ieda, 2001) and departure time choice behavior (Noland et al., 1998; Noland and Polak, 2002; Coulombel and de Palma, 2014). The variability due to departure time and route choices is associated to coordination errors due to strategic inter-actions which also contribute to the observed variability in travel time. Therefore, from a modeling standpoint it is impor-tant for models to be able to accurately predict variability in travel costs, associated to choice behavior under strategic interactions.

The increased reliance on equilibrium models in transportation to make expensive decisions leads to the fundamental question: Which equilibrium theory is behaviourally consistent? Unfortunately, field data does not provide enough control to test these equilibrium theories. A key feature of these equilibrium theories is the strategic interaction. However, most models estimated using field data (Fosgerau, 2006; Frejinger and Bierlaire, 2007; Carrion and Levinson, 2013; Dixit, 2013) consider travel times as exogenously given, without considering strategic interactions. The strategic interaction results in choices that are a consequence of feedback of the impact of the choice probabilities on path costs, which is how Stochastic User Equilibrium (SUE) is modeled in transport networks. Therefore, it is critical to understand the effects of these strategic interactions on the ultimate choice models that are used in modeling traffic.

Recently, methods from experimental economics are being used to test key theoretical aspects in transportation through controlled laboratory experiments. The key feature of experimental economics is the use of monetary incentives to create controlled environments, specifically these monetary incentives are high enough to induce the utilities and mitigate the impact of an individual's innate characteristics on the utility[1]. In this study we use several transport experiments, where indi-viduals make choices with strategic interactions and their monetary payoffs are affected by endogenous congestion. Therefore strategic interactions and equilibrium play a role in their choices.

Most of these laboratory experiments have implemented various versions of "congestion games" (Rosenthal, 1973), that consists of non-cooperative games in which a player's strategy is to choose a subset of resources, and the utility of each player only depends on the number of players choosing the same or some overlapping strategy. For instance, several studies (Ramadurai and Ukkusuri, 2007; Ziegelmeyer et al., 2008; Daniel et al., 2009) conducted lab experiments to evaluate equi-librium predictions based on a discrete version of a bottleneck model (Vickrey, 1969; Arnott et al., 1993). Differently, Morgan et al. (2009), Rapoport et al. (2006), Hartman (2012) and Denant-Boemont and Hammiche (2012) try to observe within the lab some paradoxes that could be produced by Wardrop-Nash equilibrium when users have to choose among routes (Braess Paradox) or among modes (Pigou-Knight-Downs Paradox, Downs-Thomson Paradox[2]). Anderson et al. (2008) conducted an experimental study on a Market Entry Game (MEG) to represent the traffic coordination problem, with an aim to observe how congestion charges and real-time information given to participants might improve efficiency of the system. In this vein, Denant-Boemont and Fortat (2013) studied the effect of non-linear cost functions, which are similar to the Bureau of Public Roads (BPR) function, on the equilibrium properties, the data from which is used in this study to evaluate the impact of realistic cost functions. In addition, experimental economics has also been used to study the impact of, drivers' risk aversion and sub-jective beliefs of risk on crash propensity (Dixit et al., 2014) as well as route choice (Dixit et al., 2013). Most of these studies provide strong evidence for a Mixed Strategy Nash Equilibrium (MSNE) in predicting the observed aggregate choices. However, MSNE has been found to underestimate the variances in choices.

An interesting dimension of behavior in experimental games is that even small changes in the parameters of a game, that should not theoretically change the equilibrium, can have dramatic effects on the observed level of efficiency or coordination among players (see Goeree and Holt, 2005). This study attempts to evaluate the performance of different equilibrium the-ories, particularly: Pure Strategy Nash Equilibrium, Mixed Strategy Nash Equilibrium[3] and Stochastic User Equilibrium.

This study shows that the Pure Strategy Nash (Wardrop) Equilibrium (hereafter called PSNE) is not able to explain the observed choices well. In addition, though Mixed Strategy Nash Equilibrium (hereafter denoted as MSNE) was found to explain better mean choices, however, they do not explain the variance in choices. The Stochastic User Equilibrium (SUE) was found to fit the data the most accurately. Apart from studying the performance of the various equilibrium theories com-pared to the observed data, this paper presents a method to estimate the choice model for a SUE, and understand the impact of group size as well as the complexity of the cost function on the *scale parameter* (inverse of the Fechner error) of the SUE model.

Our paper is organized as follows. The next section is dedicated to the experimental data. Section 3, provides a more detailed presentation on the method used to estimate the stochastic user equilibrium. Section 4 presents the results, which is followed by the conclusion.

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[1] This is a significant feature of experimental economics that was outlined in the seminal work on Induced Value Theory by Smith (1976, 1982)
[2] For a clear review of these paradoxes, see Arnott and Small (1994).
[3] In this paper we only deal with symmetric mixed strategy Nash equilibrium.

2. Experimental data

This study utilizes data collected from two different experiments with an aim of studying choices with strategic interaction under endogenous congestion, where subjects should either choose a mode between a public and private one, or between entering a mode/route and staying out. The first dataset was collected as part of the experiments to study the Downs-Thomson paradox (Denant-Boemont and Hammiche, 2012) as the outcome of a choice game between modes. As part of the Downs-Thomson's study, two treatments were administered with different capacities for the options. The second set of dataset was collected to study trip making with non-linear and linear cost structures, as well as with different group sizes (Denant-Boemont and Fortat, 2013). We provide more details on the experimental setup and data in the following sections.

2.1. Downs-Thomson experiment

In the Downs-Thomson experiment individuals were given a choice between using a public mode (Public Transit) or a private vehicle (Road). A participant i ($i = 1, \ldots, N$) has to make a choice δ^i between the option X (Road) and Y (Public Transit), that is δ^i indicates the choice variable, and will be X if the subject decides to take a private vehicle and use the road, or else it will be Y.

The individual payoff for a user choosing road is:

$$\pi_1 = k_1 + r_1(c_1 - m_1) \quad \text{if } \delta^i = X$$

Whereas the individual payoff of using public transit is:

$$\pi_2 = k_2 + r_2(c_2 + m_2) \quad \text{if } \delta^i = Y$$

Where c_1 and c_2 are respectively the capacities of road and public transit; m_1 and m_2 are respectively the number of users choosing road and public transit, r_1, r_2, k_1 and k_2 are positive parameters with $r_1 > r_2$ and $k_1 > k_2$. All these parameters are known by all the N players (i.e. the total number of transport users), i.e. $m_1 + m_2 = N$ and also $c_1 + c_2 < N$.

A Pure Strategy Nash Equilibrium occurs when the payoff's on the two options are equal, i.e. when

$$m_1 = (k_1 - k_2 - r_2(N + c_2) + r_1 c_1)/(r_1 - r_2) \text{ and } m_2 = N - m_1$$

There is also a Symmetric Mixed-Strategy Nash equilibrium, obtained by equating expected payoffs (E_1, E_2) for Road and Public Transit and by considering that, for a given user, the number of entrants on Road is given by $m_1 = p_1(N - 1) + 1$, and $p_2 = 1 - p_1$. Solving for p_1 we get:

$$E_1(p_1, N) = E_2(1 - p_1, N)$$

$$k_1 + r_1(c_1 - p_1(N - 1) - 1) = k_2 + r_2(c_2 + (1 - p_1)(N - 1))$$

The parameters used for the Downs-Thomson experiment, as well as the theoretical predictions for entry rates at Pure Strategy Nash Equilibriums (PSNE) and probability to enter on road at MSNE (Mixed-Strategy Nash Equilibrium), are reported in Table 1. As presented in the table, there were 2 experimental treatments consisting of two different capacity levels for road (LOW with $c_1 = 3$ or HIGH with $c_1 = 6$). The original aim of this experiment was to observe theoretical predictions of the Downs-Thomson Paradox in the Lab, i.e., increase in capacity (HIGH treatment) results in decrease in users' payoff.

The experimental sessions were held in LABEX-EM, Rennes using the ZTREE platform (Fischbacher, 2007). The average duration of a session was 1.5 h with an average payoff of 15 euros. For a given experimental session, a group of 14 subjects participated in this experiment. The experiment involved 8 sessions with a total of 112 subjects. Each session had a total of 40 rounds, 20 rounds with a low capacity treatment and 20 rounds with high capacity treatment.

Table 1
Parameters for the Downs-Thomson's experiments with various treatments.

	Low capacity treatment	High capacity treatment
k_1	6	6
k_2	0	0
r_1	1	1
r_2	0.25	0.25
c_1	3	6
c_2	1	1
PSNE	$m_1 = 6$	$m_1 = 10$
MSNE	$p_1 = 0.47$	$p_1 = 0.77$

Table 2
Experimental treatments.

Treatment	Experiment type	Cost function	Group size	Number of participants
Quadratic MEG12	Market entry game	Non-linear	12	120
Quadratic MEG24	Market entry game	Non-linear	24	240
Linear MEG12	Market entry game	Linear	12	120
Linear MEG24	Market entry game	Linear	24	240
DT_Cap3	Down-Thomson/capacity 3	Linear	14	112
DT_Cap6	Down-Thomson/capacity 6	Linear	14	112

2.2. Snowball experiment

The snowball experiment was setup to study User Equilibrium (UE) predictions on whether people choose to make trips or not, and its sensitivity to different (realistic or un realistic) cost functions. The experimental treatments were designed by modeling Market (transport network) Entry Games (hereafter referred to as MEG; see Selten and Guth, 1982; Gary-Bobo, 1990). In this game, a participant i (i = 1, …, N) chooses between entering the transport network ($\delta^i = 1$) or staying out ($\delta^i = 0$). The experiment involved two treatments with regard to specification of the cost function, one cost function was linear (and identical to the one implemented by Anderson et al., 2008), and from here on is referred to as linear market entry games or Linear MEG for short. The second experiment involved a realistic quadratic cost function that is similar to the BPR travel time function, which will be referred to as quadratic market entry game or quadratic MEG for short. Further, there were two additional treatments with regard to group size one with 12 participants and the other with 24 participants.

The pay off function corresponding to the linear MEG studied by Anderson et al. (2008):

$$\pi_L(\delta^i) = \begin{cases} 0.5 & \text{if } \delta^i = 0 \\ 0.5 + 0.5(8 - m) & \text{if } \delta^i = 1 \end{cases}$$

Where π_L is the payoff for a Linear MEG; δ^i is the decision to make the trip (δ^i = 1) or not (δ^i = 0) and m are the total participants choosing to make the trip, where $m \leqslant N$, N being the total number of participants. The PSNE for the linear MEG would be when m = 7 or m = 8. The MSNE is calculated by equating the expected payoffs of the option of making the trip $E_L(\delta^i = 1)$ or staying $E_L(\delta^i = 0)$, where users attribute a probability (p_L) for individuals to choose making a trip, i.e.

$$E_L(\delta^i = 1, p_L) = E_L(\delta^i = 0, 1 - p_L)$$

$$0.5 + 0.5(8 - p_L(N - 1) - 1) = 0.5$$

In the quadratic MEG, the pay off function corresponds to:

$$\pi_Q(\delta^i) = \begin{cases} 0.5 & \text{if } \delta^i = 0 \\ 4.5 - 4\left(\frac{m}{8}\right)^2 & \text{if } \delta^i = 1 \end{cases}$$

Where, π_Q is the payoff for a quadratic MEG; δ^i is the decision to make the trip (δ^i = 1) or not (δ^i = 0) and m are the total participants choosing to make the trip, where $m < N$. N is the total number of participants. A Pure Strategy Nash Equilibrium for the quadratic MEG is reached when m = 7 or m = 8[4]. The Mixed Strategy Nash Equilibrium is calculated by equating the expected payoffs $E_Q(\delta^i = 1)$ of the options, where users attribute a probability (p_Q) for individuals who choose to make a trip, i.e.

$$E_Q(\delta^i = 1, p_Q) = E_Q(\delta^i = 0, 1 - p_Q)$$

$$4.5 - 4\left((p_t^Q(N - 1) - 1)/8\right)^2 = 0.5$$

Under Mixed Strategy Nash Equilibrium, for linear MEG as well as for quadratic MEG, the probability to enter on road is respectively 0.64 for N = 12 and 0.3 for N = 24.

The sessions for these experiments were conducted in the LABEX, University of Rennes 1. There were 360 participants. The average pay off was around 15 euros, for an average total duration of 1.5 h. All sessions were computerized using the ZTREE software (Fischbacher, 2007). Participants were invited to sessions by using ORSEE software (Greiner, 2004).

The various treatments and experiments are summarized in Table 2. The effects of the complexity of the cost function and number of subjects on the choices in equilibrium were inferred from the treatment effects.

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[4] As the capacity (denominator of Eq. (13), which is 8) is an integer, there are many PSNE compatible with m = 8 or m = 7 (Erev and Rapoport, 1998).

3. Estimation of Choice Model for Stochastic User Equilibrium

A Stochastic User Equilibrium (SUE) is an equilibrium in which travelers take the route or mode with the lowest perceived costs (Daganzo and Sheffi, 1977)[5]. When the random error term of the perceived costs for the alternatives satisfies a Gumbel (Normal) distribution, then the choice probability can be described as a logit (probit) model. In this paper, we use a logit model, where the number of choices for an alternative i (f_i) in a choice set I may be written as:

$$f_i = \frac{e^{-\theta c_i}}{\sum_{i' \in I} e^{-\theta c_{i'}}} D \quad , \forall i \in I \tag{1}$$

Where, D is the total demand. c_i is the cost of alternative i which is a function of f_i. θ is a scale parameter, which predicts completely random behavior (probability of 0.5), if $\theta = 0$ and deterministic behavior if $\theta = \infty$. It is important to point out here that this scale parameter θ can be interpreted as the reciprocal of Fechner error (μ) in the Fechner error specification (Andersen et al., 2010).

$$f_i = \frac{e^{-c_i/\mu}}{\sum_{i' \in I} e^{-c_{i'}/\mu}} D \quad , \forall i \in I \tag{2}$$

Due to the psychological interpretations of Fechner error (further discussed in Andersen et al., 2010), we use the equivalent Fechner error formulation in our study.

Variability in travel time is an important consideration in mode or route choice models, and as aptly characterized by Watling (2006)

"Classical stochastic user equilibrium (SUE) methods allow the mean travel times to be differentially perceived across the population, yet in a conventional application neither the UE or SUE approach recognizes the travel times to be inherently variable."

This issue raised by Watling (2006) can be easily addressed by replacing the cost (c_i) with the expected cost ($E[c_i]$) on the path and assuming that people behave stochastically in their choices with a Fechner Error Specification.

$$f_p^{rs} = \frac{e^{-E[c_i]/\mu}}{\sum_{i' \in I} e^{-E[c_{i'}]/\mu}} D \quad , \forall i \in I \tag{3}$$

As mentioned earlier, in order to maintain control a simple two link network with one origin destination pair was used in the experiment. The simplicity of the experiment provided laboratory control and helped the experimenters avoid confounding effects of the complexity of choice in networks (which result in combinatorial number of paths), the effect of overlapping paths (which would result in violation of I.I.A). The confounding effects are interesting and important questions, however the focus of this paper is a fundamental one, i.e., what type of equilibrium best explains choices in endogenous congestion, and how is the Fechner error (or the scale factor) affected by the level of demand (number of participant) and the complexity of the cost function. In lieu of this, all the experiments analyzed are a binary choice problem with strategic interaction and endogenous congestion. Which reduces Eq. (3) to:

$$P_i = \frac{e^{-E[c_i(P_i)]/\mu}}{e^{-E[c_{i'}(P_{i'})]/\mu} + e^{-E[c_i(P_i)]/\mu}} \quad , \forall i \in I \tag{4}$$

In this particular case, since the expected costs for an alternative depends on the number of people choosing an alternative due to the strategic interactions, the expected costs are function of the probability of participants choosing that alternative, which is shown in Eq. (4).

This particular study uses methods from experimental economics, and as discussed earlier due to Induced Value Theory (Smith, 1976, 1982) the utilities are values induced through known monetary incentives, however, the Fechner Error (or scale parameter) is unknown and needs to be estimated. This is done by exploiting the SUE condition that the choice probabilities (P_i) used in calculating the expected costs ($E[c_i(P_i)]$) are the same as that estimated by the logit model in Eq. (4).

As mentioned in the introduction, most models estimated using field data (Fosgerau, 2006; Frejinger and Bierlaire, 2007) consider travel times as exogenously given, without considering strategic interactions, and therefore have not needed to rely on meeting the above mentioned SUE condition. Based on our extensive literature search in the transportation field, the model estimation proposed here is the first of its kind.

In the congestion games discussed in this study, choices are binary and the perceived expected travel costs $E\pi = (E\pi_1, E\pi_2)$ are:

$$E\pi_1 = f(N, P) \tag{5}$$

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[5] In many ways, SUE is very similar to the Quantal Response Equilibrium approach (see McKelvey and Palfrey, 1995, 1998; Goeree et al., 2008).

$$E\pi_2 = g(N, 1 - P) \tag{6}$$

Where, N is the total number of participants and P the probability of choosing option one, and $1 - P$ being the probability of choosing the other option. f and g are expected cost functions for the respective choices.

As discussed earlier, SUE allows for subjects to make some *behavioral* errors associated to perception. The notion of error is one that uses a non-degenerate probability link function between the latent index $\nabla E\pi$ and the probability of picking a specific option, which also converges to or Pure Nash Equilibrium. We employ the error specification originally due to Fechner and popularized by Hey and Orme (1994). This error specification posits the latent index

$$\nabla E\pi = (E\pi_1 - E\pi_2)/\mu \tag{7}$$

Where, μ is a structural "noise parameter" used to allow some errors from the perspective of the deterministic Expected Utility Theory (EUT) model. As $\mu \to 0$ this specification collapses to the deterministic choice EUT model, where the choice is strictly determined by the Expected Utilities of the two options. Therefore as $\mu \to 0$ at equilibrium there should be strict equality between $E\pi_1$ and $E\pi_2$, which is the condition for PSNE. But as μ gets larger the choice essentially becomes random. Thus μ can be viewed as a parameter that flattens out the link functions, as it gets larger. As discussed earlier, this noise parameter is the inverse of the scale parameter used in logit models (Ben-Akiva and Morikawa, 1990; Bradley and Daly, 1994).

By varying the shape of the link function, one can imagine subjects that are more (or less) sensitive to a given difference in the latent index ($\nabla E\pi$).

$$P = \varphi(\nabla E\pi) \tag{8}$$

Assuming that φ is a logistic distribution implies that $\boldsymbol{P}$ can be represented by a Logit formulation, which is the same as Eq. (4) for binary choice with strategic interaction:

$$P = \frac{e^{\left(\frac{E\pi_1 - E\pi_2}{\mu}\right)}}{1 + e^{\left(\frac{E\pi_1 - E\pi_2}{\mu}\right)}} \tag{9}$$

Replacing Eqs. (5) and (6) we get

$$P = \frac{e^{\left(\frac{f(N,P) - g(N,1-P)}{\mu}\right)}}{1 + e^{\left(\frac{f(N,P) - g(N,1-P)}{\mu}\right)}} \tag{10}$$

The likelihood of the observed responses, conditional on the assumption of risk neutrality, depends on the estimates of μ given the above statistical specification and the observed choices. The "statistical specification" here includes assuming some functional form for the Cumulative Density Function (CDF). The conditional log-likelihood is then

$$\ln L(\mu; y_i, \mathbf{X}) = \sum_i [\ln \varphi(\nabla E\pi) \times \mathbf{I}(y_i = 1) + \ln(1 - \varphi(\nabla E\pi)) \times \mathbf{I}(y_i = -1))] \tag{11}$$

Where $\mathbf{I}(\cdot)$ is the indicator function, $y_i = 1(-1)$ denotes that subject i chose one option or the other, and $\mathbf{X}$ is a vector of different experiments and treatments.

It is also a simple matter to generalize the Maximum Likelihood (ML) analysis to allow the core parameter μ to be a linear function of dummy variables for each experimental treatment. The model is extended by specifying: $\mu = \mu_0 + BX$, where μ_0 is a fixed parameter and B is a vector of effects associated with each characteristic in the variable vector $\mathbf{X}$. In effect, the unconditional model would assume $\mu = \mu_0$ for all the treatments. This extension significantly enhances the richness of structural ML estimation, particularly for responses pooled over different subjects and treatments.

Based on Eq. (10), it can be concluded that statistically computing $\boldsymbol{P}$ is a fixed point problem of the type $P = H(P)$, where H could be any function. The probability $\boldsymbol{P}$ is a solution of a SUE (Eq. (10)), which should coincide with the choice probability given by the logit formula (Eq. (10)), where μ needs to be estimated using maximum likelihood (Eq. (11)). Such a fixed-point problem is usually solved iteratively, using estimates from previous iteration, as inputs for the next, until convergence is achieved. To solve this, μ is estimated iteratively using Eq. (11). To utilize Eq. (11) in the iterative estimation process, it is rewritten as:

$$\ln L(\mu_{i+1}; y, \mathbf{X}) = \sum_i [\ln \varphi(\nabla E\pi(P_i)) \times \mathbf{I}(y = 1) + \ln(1 - \varphi(\nabla E\pi(P_i))) \times \mathbf{I}(y = -1))] \tag{12}$$

As shown in Eq. (12), μ_{i+1} is estimated based on probabilities P_i calculated in the previous iteration. The new μ_{i+1} is then used to estimate the new probabilities, which is shown in Eq. (13)

$$P_{i+1} = \frac{e^{\left(\frac{f(N,P_i) - g(N,1-P_i)}{\mu_{i+1}}\right)}}{1 + e^{\left(\frac{f(N,P_i) - g(N,1-P_i)}{\mu_{i+1}}\right)}} \tag{13}$$

The estimated probabilities are used as inputs in the next iteration for functions f and g to estimate the noise parameter (μ). These Fechner estimates are then used to generate the probabilities for the next iteration. We solve the above series of equations from 12 to 13 till the probability estimates for successive iterations converge to be the same.

In the next section, we implement this and study the performance of the SUE. In addition, we investigate the impact of the different experiments on the noise parameter. The next section describes the experiments and the data used for analysis.

4. Results

This section presents the results of the SUE estimates, and attempt to explain the scale and the signs of the noise parameter estimated for the various experimental treatments. Finally, the study compares the observed equilibrium choice probabilities with those estimated from the SUE, as well as with the theoretical predictions of PSNE and MSNE.

4.1. Estimates of Stochastic User Equilibrium

Table 3 presents the estimates of the noise parameter (μ). In fact, the fixed-point problem described in Eq. (9) was solved relatively quickly, using a starting value of 0.5 for each of the probabilities. The probabilities and estimates of the Fechner coefficient converged in two iterations.

An important aspect to note in Table 3 is the possibility of multiple solutions in the case of DT_Cap3. Reconsidering Eq. (9), if $E\pi_1 = E\pi_2$ then the choice probabilities (P) are 0.5 for any value of μ, and therefore μ is not identifiable. On the contrary, if $E\pi_1 \neq E\pi_2$, then P is 0.5 as $\mu \to \infty$. Therefore, conducting experiments where choice probabilities for options are equi-probable is problematic from an estimation stand point, and therefore should be avoided in an experimental design setup.

Devetag and Ortmann (2007) in a seminal review of coordination games and experiments in game theory found that group size and the complexity of the game plays an important role in co-ordinating to equilibrium. The result for the estimates of μ presented in Table 3 provides more evidence with respect to the findings by Devetag and Ortmann (2007).

It is evident from Table 3, that there is a significant difference between the small and large group size on the noise parameter. It is observed that in the case of quadratic and linear market entry game μ is greater in the case when the number of participants is greater, i.e., the coefficient for *quadraticMEG12* is less than *quadraticMEG24*, and the coefficient of *linearMEG12* is less than *linearMEG24*. This implies that the behavioral noise parameter (μ) is larger when group size is higher. This is perhaps due to greater effort needed by the participants to coordinate, and identified by Devetag and Ortmann (2007). However, it is encouraging to see that this phenomenon is well captured by the SUE model.

Systematic differences were also found between the quadratic and linear market entry games. For a given group size the linear market entry games were found to have smaller values of μ (i.e. coefficients for *quadraticMEG12 > linearMEG12*; *quadraticMEG24 > linearMEG24*), indicating that the behavioral noise parameter (μ) was larger when the costs are quadratic for a given group size. Again, this could be attributed to the coordination problems when cost functions are more complex in a non-linear case as compared to the linear case that would require more effort.

The fact that the complexities of the cost functions and demand levels impact the noise parameter indicates that the variance in choices are affected by these factors, and neglecting these effects would provide biased estimates of the choice probabilities, and need to be further explored with field data.

4.2. Comparison of equilibrium theory

Based on the estimated model presented in Table 3, the probabilities for the SUE were estimated using Eq. (9). In the case of the Linear MEG and Quadratic MEG the probability of choosing to make the trip was calculated, while in the Down-Thomson paradox experiments the probabilities of choosing to take the road were calculated. The estimated probabilities were used to determine the variance and the expected number of subject making choices. The expected number was simply the product of the number of subjects and the probability of making the choice (Np). The standard deviation in number of subjects making the choice was calculated as $\sqrt{Np(1-p)}$ (see Rapoport, 1995).

Table 3
Estimates of Noise parameter.

Experimental treatment	Coeff. μ	Std. Err.	z	$P > |z|$
Quadratic MEG12 (N = 12, Quadratic MEG)	0.946	0.065	14.554	0.000
Quadratic MEG24 (N = 24, Quadratic MEG)	3.223	0.234	13.774	0.000
Linear MEG12 (N = 12, Linear MEG)	0.149	0.012	12.417	0.000
Linear MEG24 (N = 24, Linear MEG)	0.928	0.08	6.739	0.000
DT_Cap3 (Capacity = 3, Down-Thomson)	Degenerate case with multiple solutions			
DT_Cap6 (Capacity = 6, Down-Thomson)	0.908	0.042	21.619	0.000

Table 4
Comparison of observed equilibrium with PSNE, MSNE and SUE.

	Twelve (# observations = 200*)			Twenty Four (# observations =100*)		
Linear MEG	**Prob**	**Expected**	**Stdev**	**Prob**	**Expected**	**Stdev**
Observed	0.66	7.89	1.71	0.37	8.61	2.05
Pure Strategy Nash (Wardrop) Equilibrium	0.00	8.00	0.00	0.00	8.00	0.00
Mixed Strategy Nash Equilibrium	0.64	7.64	1.66	0.30	7.30	2.24
Stochastic User Equilibrium	0.66	7.89	1.64	0.36	8.61	2.35
	Twelve (# observations = 400*)			Twenty Four (# observations =200*)		
Quadratic MEG	**Prob**	**Expected**	**Stdev**	**Prob**	**Expected**	**Stdev**
Observed	0.64	7.69	1.71	0.37	8.91	2.49
Pure Strategy Nash (Wardrop) Equilibrium	0.00	8.00	0.00	0.00	8.00	0.00
Mixed Strategy Nash Equilibrium	0.64	7.68	1.66	0.30	7.30	2.25
Stochastic User Equilibrium	0.64	7.68	1.66	0.37	8.90	2.37
	Capacity=3 (# observations = 160*)			Capacity=6 (# observations = 160*)		
Downs-Thomson	**Prob**	**Expected**	**Stdev**	**Prob**	**Expected**	**Stdev**
Observed	0.50	7.00	2.03	0.73	10.25	1.58
Pure Strategy Nash (Wardrop) Equilibrium	0.00	6.00	0.00	0.00	10.00	0.00
Mixed Strategy Nash Equilibrium	0.47	6.51	1.87	0.77	10,78	1.59
Stochastic User Equilibrium	0.50	7.00	1.87	0.73	10.25	1.66

*The numbers relate to equilibrium observations at the aggregate level.

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Table 5
p-values for the z-test between observed data with MSNE and SUE.

	quadraticMEG12	quadraticMEG24	linearMEG12	linearMEG24	DT_Cap3	DT_Cap6
MSNE	0.000	0.000	0.771	0.000	0.006	0.141
SUE	1.000	1.000	0.771	0.932	1.000	0.952

The probabilities, expected numbers and variances were calculated based on the observed data in the experiments. These observed values are presented in Table 4 with the SUE estimates as well as the theoretical predictions for the PSNE and MSNE.

As observed in previous studies (Daniel et al., 2009; Rapoport et al., 2006; Morgan et al., 2009; Selten et al., 2007; Ziegelmeyer et al., 2008; Denant-Boemont and Hammiche, 2012) the observed experimental data did not fit the predictions for the PSNE well. Critically, the Pure Strategy Nash Equilibrium did not predict any variance in aggregate choices that were observed in the data. The Mixed Strategy Nash Equilibrium concept performed much better and its theoretically predicted equilibrium as well as its variances were much closer to the observed data. The predictions of the estimated SUE were even better compared to MSNE, especially in the cases when group size is large under the quadratic Market Entry Game experiment as well as for the Downs-Thomson experiment where the road capacity was set equal to six.

A z-test was carried out between the predictions of MSNE and SUE with the experimentally observed data for the various experiments. The PSNE is not included since it does not predict any variances in choices. The green shaded cells indicate no significant statistical difference between observed data and the theory. The pink area indicates statistically significant different at a 95% confidence level. The yellow shaded cell for DT_cap6 indicates that there is a significant difference at an 85% confidence level. The p-values for the z-test are shown in Table 5. It is observed that the SUE model is able to predict the observed data much better than MSNE.

When attempting to understand uncertainty in traffic, it is important to recognize which route choice models (PSNE, MSNE or SUE) replicate strategic traffic interactions realistically. Therefore, it is critical to point out that though the absolute differences between the observed and those predicted by PSNE and MSNE might be small, these differences are statistically significant. However, SUE predicts the observed strategic choices well.

5. Conclusion

The aim of this paper was to assess the predictive abilities of well-known behavioral equilibrium theories using data collected from experimental games. An important finding of this study is that explicitly accounting for behavioral errors through Stochastic User Equilibrium can significantly improve equilibrium predictions. In particular, stochastic traffic equilibrium was found to more accurately predict observed mean choices as well as the variances in the accompanying the equilibrium. Based on various experimental data, the predictions associated to Stochastic User Equilibrium were found to perform better than the Pure Nash Strategy equilibrium or Mixed Strategy Nash Equilibrium predictions. This has important implications for transportation network models that want to evaluate impact of route choice behavior on the reliability of network (Mirchandani and Soroush, 1987; Uchida and Iida, 1993; Lo and Tung, 2000; Bell, 2000; Yin and Ieda, 2001). In this study, the SUE model was found to accurately predict the variability in route choice behavior.

From an experimental design standpoint having probabilities that are equal between different alternatives, results in multiple solutions for the noise parameter associated to the Fechner error. Therefore it is not possible to estimate it. Hence, to analyze impact of experimental treatments on the noise parameter and Fechner error, it is critical to design experiments where the choice probabilities are not equal between different alternatives.

Additionally, this study outlines a method to estimate a SUE, where we allow the coefficient identified for noise parameter (μ) to be heterogeneous. The coefficient for the noise parameter (μ) provides insights into factors that cause behavioral errors. Especially, in game theoretic equilibrium models, co-ordination plays an important role in determining the equilibrium. Based on this study, evidence was found that larger group sizes and higher amount of effort result in larger behavioral errors and more noisy decisions due to lesser coordination. This results in different equilibriums than those predicted by pure and mixed Strategy Nash equilibriums concepts.

Critically, the impact of group size and complexity of the cost function on the noise parameter may have implications on transferability of demand models. Based on this finding, it seems that as the total demand increases or complexity of the cost function increases the noise parameter would increase, which implies that the scaling factor would decrease. This suggests a need for a theory for effect of group size and complexity of cost on the noise parameter. The impact of the effect of the complexity of the payoff function on the Fechner noise parameter also indicates the need to use realistic payoff functions in experiments to ensure a certain degree of external validity.

Finally, there is further research required with more controlled experiments to tease out effects of context sensitivity, risk or loss aversion, probability weighting and beliefs on the noise parameter (μ) and providing better equilibrium predictions. A more significant evaluation is needed with respect to larger networks and with overlapping routes.

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References

Anderson, L., Holt ,C.A., Reiley, D., 2008. Congestion pricing and welfare: an entry experiment. In: Cherry, T., Koll, T., Shogren, J. (Eds.), Environmental Economics, Experimental Methods, Chapter 15, pp. 280–92.
Andersen, S., Harrison, G., Lau, M., Rutström, E., 2010. Behavioral econometrics for psychologists. J. Econ. Psychol. 31, 553–576.
Arnott, R., De Palma, A., Lindsey, R., 1993. A structural model of peak-period congestion: a traffic bottleneck with elastic demand. Am. Econ. Rev. 83, 161–179.
Arnott, R., Small, K., 1994. The economics of traffic congestion. Am. Sci. 82, 446–455.
Bell, M.G.H., 2000. A game theory approach to measuring the performance reliability of transport networks. Transp. Res. Part B 34 (6), 533–545.
Ben-Akiva, M., Morikawa, T., 1990. Estimation of switching models from revealed preferences and stated intentions. Transp. Res. Part A 24, 485–495.
Bradley, M., Daly, A., 1994. Use of the logit scaling approach to test for rank-order and fatigue effects in stated preference data. Transportation 21, 167–184.
Carrion, C., Levinson, D., 2013. Valuation of travel time reliability from a GPS-based experimental design. Transp. Res. Part C: Emerg. Technol. 35, 305–323.
Coulombel, N., de Palma, A., 2014. The marginal social cost of travel time variability. Transp. Res. Part C: Emerg. Technol., Available online 11 January 2014, ISSN 0968-090X.
Daganzo, C.F., Sheffi, Y., 1977. On stochastic models of traffic assignment. Transp. Sci. 11 (3), 253–274.
Daniel, T.E., Gisches, E.J., Rapoport, A., 2009. Departure time in Y-shaped traffic networks with multiple bottlenecks. Am. Econ. Rev. 99 (5), 2149–2176.
Denant-Boemont, L., Hammiche, S., 2012. Downs-Thomson Paradox and Public Transit Capacity in the Laboratory. ITLS WP-12-10.
Denant-Boemont, L., Fortat, V., 2013. Snowball Effect and Traffic Equilibrium in a Market Entry Game: A Laboratory Experiment. Transportation Research Board Annual Meeting 2013, Washington D. C.
Devetag, G., Ortmann, A., 2007. When and why? A critical survey on coordination failure in the laboratory. Exp. Econ. 10 (3), 331–344.
Dixit, V.V., Harrison, G.W., Rutstrom, E.E., 2014. Estimating the subjective risks of driving simulator accidents. Accid. Anal. Prev. 62 (2014), 63–78.
Dixit, V. V., Harb, R., Martinez, J., Rutstrom, E.E., 2013. Eliciting risk attitudes from route choices over simulated driving choices. Transportation Research Board Annual Meeting 2013, Washington D. C.
Dixit, V.V., 2013. Behavioural foundations of two-fluid model for urban traffic. Transp. Res. Part C: Emerg. Technol. 35, 115–126, October 2013.
Erev, I., Rapoport, A., 1998. Coordination, "Magic", and reinforcement learning in a market entry game. Games Econ. Behav. 23, 146–175.

Fischbacher, U., 2007. Z-Tree: Zurich toolbox for ready-made economic experiments. Exp. Econ. 40 (2), 171–178.

Fosgerau, M., 2006. Investigating the distribution of travel time savings. Transp. Res. Part B 40 (8), 688–707.

Frejinger, E., Bierlaire, M., 2007. Capturing correlation with subnetworks in route choice models. Transp. Res. Part B 41 (3), 363–378.

Gayah, V.V., Dixit, V.V., Guler, S.I., 2013. Relationship between mean and day-to-day variation in travel time in urban networks. EURO J. Transp. Logist. Available from: <http://dx.doi.org/10.1007/s13676-013-0032-2>.

Gary-Bobo, R., 1990. On the existence of equilibrium points in a class of asymmetric market entry games. Games Econ. Behav. 2, 239–246.

Goeree, J.K., Holt, C.A., 2005. An explanation of anomalous behavior in models of political participation. Am. Political Sci. Rev. 99 (2), 201–213.

Goeree, J.K., Holt, C.A., Palfrey, T., 2008. Quantal response equilibrium. In: New Palgrave Dictionary of Economics, Second Edition, Editors Steven N. Durlauf and Lawrence E. Blume, Palgrave Macmillan.

Greiner, B., 2004. An online recruitment system for economic experiments (ORSEE). In: Kurt Kremer, Volker Macho (Hrsg.): Forschung und wissenschaftliches Rechnen. GWDG Bericht 63.Ges.frWiss. Datenverarbeitung, Gottingen, pp. 79–93.

Hanson, S., Huff, J., 1988. Systematic variability in repetitious travel. Transportation 15, 111–135.

Hartman, J., 2012. A route choice experiment with an efficient toll. Networks Spatial Econ. 12 (2), 205–222.

Hey, J.D., Orme, C.D., 1994. Investigating generalisations of expected utility theory using experimental data. Econometrica 62 (6), 1291–1326.

Jha, M., Madanat, S., Peeta, S., 1998. Perception updating and day-to-day travel choice dynamics in traffic networks with information provision. Transp. Res. Part C: Emerg. Technol. 6 (3), 189–212.

Kim, J., Mahmassani, H.S., 2014. A finite mixture model of vehicle-to-vehicle and day-to-day variability of traffic network travel times. Transp. Res. Part C: Emerg. Technol. 46, 83–97.

Lo, H., Tung, Y.K., 2000. A chance constrained network capacity model. In: Bell, M., Cassir, C. (Eds.), Reliability of Transport Networks. Research Studies Press Ltdr, pp. 159–172.

McKelvey, R., Palfrey, T., 1995. Quantal response equilibria for normal form games. Games Econ. Behav. 10, 6–38.

McKelvey, R., Palfrey, T., 1998. Quantal response equilibria for extensive form games. Exp. Econ. 1, 9–41.

Mirchandani, P., Soroush, H., 1987. Generalized traffic equilibrium with probabilistic travel times and perceptions. Transp. Sci. 21 (3), 133–152.

Morgan, J., Orzen, H., Sefton, M., 2009. Network architecture and traffic flows: experiments on the Pigou-Knight-Downs and Braess Paradoxes. Games Econ. Behav. 66, 348–372.

Noland, R.B., Small, K.A., Koskenoja, P.M., Chu, X., 1998. Simulating travel reliability. Reg. Sci. Urban Econ. 28, 535–564.

Noland, R.B., Polak, J.W., 2002. Travel time variability: a review of theoretical and empirical issues. Trans. Rev. 22 (1), 39–54.

Pas, E.I., Sundar, S., 1995. Intra-personal variability in daily urban travel behavior: some additional evidence. Transportation 22, 135–150.

Rapoport, A., 1995. Individual strategies in a market entry game. Group Decis. Negot. 4, 117–133.

Selten, R., Guth, W., 1982. Equilibrium point selection in a class of market entry games. In: Diestler, M., Furst, E., Schwadiauer, G. (Eds.), Games, Economic Dynamics and Time Series Analysis. Physica Verlag, Berlin, pp. 101–116.

Smith, V.L., 1976. Experimental economics: induced value theory. Am. Econ. Rev. 66 (2), 274–279.

Smith, V.L., 1982. Microeconomic systems as an experimental science. Am. Econ. Rev. 72 (5), 923–955.

Stopher, P., Kockleman, K., Greaves, S.P., Clifford, E., 2008. Reducing burden and sample sizes in multiday household travel surveys. Transp. Res. Rec.: J. Transp. Res. Board 2064, 12–18.

Ramadurai, G., Ukkusuri, S.V., 2007. Dynamic traffic equilibrium: theoretical and experimental network game results in the single bottleneck model. Transp. Res. Rec.: J. Transp. Res. Board 2029, 1–13.

Rapoport, A., Mak, V., Zwick, R., 2006. Navigating congested networks with variable demand: experimental evidence. J. Econ. Psychol. 27, 648–666.

Rosenthal, R.W., 1973. A class of games possessing pure-strategy Nash equilibria. Int. J. Game Theory 2, 65–67.

Selten, R., Chimura, T., Pitz, T., Kube, S., Schreckenberg, M., 2007. Commuters route choice behavior. Games Econ. Behav. 58 (2), 394–406.

Uchida, T., Iida, Y., 1993. Risk assignment: a new traffic assignment model considering risk of travel time variation. In: Daganzo, C.F. (Ed.), Proceedings of the 12th International Symposium on Transportation and Traffic Theory, Amsterdam, pp. 89–105.

Vickrey, W., 1969. Congestion theory and transport investment. Am. Econ. Rev. 59, 251–260.

Watling, D.P., 2006. User equilibrium traffic network assignment with stochastic travel times and late arrival penalty. Euro. J. Oper. Res. 175 (3), 1539–1556.

Wardrop, J., 1952. Some theoretical aspects of road traffic research. Proc. Inst. Civil Eng. Part I (1), 325–378.

Yin, Y., Ieda, H., 2001. Assessing performance reliability of road networks under non-recurrent congestion. Transp. Res. Rec. 1771, 148–155.

Ziegelmeyer, A., Koessler, K., BounMy, K., Denant-Boemont, L., 2008. Road traffic congestion and public information. J. Trans. Econ. Policy 42 (1), 43–82.