# A Path Enumeration Algorithm for Strategic System Optimal Dynamic Traffic Assignment

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*Submitted to the Dynamic Traffic Assignment 2014 Symposium*

**Abstract**

**Accounting for sources of uncertainty in the Dynamic Traffic Assignment (DTA) problem makes an already complex problem even more difficult. This work examines a Strategic System Optimal DTA (StrSODTA) model where optimal path proportions are assigned to minimize the Total System Travel Time (TSTT) across a range of stochastic demand scenarios. However, the LP proposed by Waller et al. (2013) faces significant challenges in terms of computational complexity. This works aims to propose a novel solution algorithm to improve the computational efficiency of the LP formulation for the StrSODTA problem. Namely, we develop a tailored path enumeration strategy to reduce the number of constraints in the aforementioned LP formulation and derive a lower bound on the optimal expected TSTT to measure the progress of the solution algorithm. Results on two benchmark traffic assignment networks are examined and we report that the proposed solution algorithm is able to significantly improve the tractability of the StrSODTA problem.**

# 1 Introduction

We address the System Optimum (SO) Dynamic Traffic Assignment (DTA) problem where travel demand is a random variable and travellers make strategic decisions to minimize expected system cost. The SO variant of the DTA is focused on finding the optimal assignment of traffic volumes when system performance is optimized. In general, the Total System Travel Time (TSTT) is used as a performance metric for SODTA. Due to the nonlinear formulation of the DTA problem, most approaches are built on simplifications and are generally difficult to solve when considering robust traffic flow models (i.e. those which possess shockwave propagation, link spillover, queue formation/dissipation, etc.). Ziliaskopoulos (2000) proposed an innovative approach to address this problem by using a Linear Programming (LP) model to solve the SODTA problem. This LP formulation uses the Cell Transmission Model (CTM) introduced by Daganzo (1994, 1995) — a well-known traffic flow model consistent with the hydrodynamic paradigm — to propagate traffic flow in the network and is developed on the rationale that the volume of traffic in each cell in the network is directly related to its travel time. This formulation enables the vast literature on LP to be used to solve the SODTA problem as well as stochastic variants and bi-level optimization approaches.

The complexity of the underlying optimization problem increases dramatically when the stochastic nature of transportation networks, e.g. travel time variability or volatility, is taken into consideration. Strategic assignment aims to equilibrate an expected cost which accounts for several stochastic demand scenarios, each of which representing a possible realization of the travel demand on the network. The different stochastic demand scenarios can be interpreted as different states of the network, representing for instance different days of the week. In this work, the probability that each stochastic demand scenario is realized is assumed to be known a priori and the associated DTA problem aims to find the optimal proportion of path flows for each Origin-Destination (OD) pair across all stochastic demand scenarios. Waller et al. (2013) recently proposed a novel approach to DTA which expands the SODTA of Ziliaskopoulos (2000) with the strategic assignment paradigm. This work developed a LP formulation for the Strategic System Optimal Dynamic Traffic Assignment (StrSODTA) which is able to account for the realization of multiple stochastic demand scenarios in the network and path choice strategies. However, this LP formulation requires an exponential number of constraints due to the large number of paths existing between any OD pair in the network. Consequently, solving the StrSODTA on realistic transportation networks is likely to turn out impractical due to space and time limitations.

The main contribution of this study is to propose a problem reduction method to improve the computational efficiency of the LP model proposed by Waller et al. (2013) to solve the StrSODTA problem. Namely, we develop a tailored path enumeration strategy to reduce the number of constraints in the aforementioned LP model. Our approach is based on the rationale that not all paths are eventually going to be used at the optimum assignment of traffic volumes. In particular, the shortest time-paths for each OD pair are the most likely to be used.

The paper is organized as follows: the literature on SODTA and strategic assignment is described in Section 2, the StrSODTA problem is presented in Section 3 and the proposed solution method is proposed in Section 4. Section 5 details the results obtained on different tests networks and concludes the paper.

## 2 Literature Review

The LP formulation utilized in this work has its foundation in the model proposed by Ziliaskopoulos (2000) that incorporates Daganzo's CTM (Daganzo, 1994, 1995) into a single origin, single destination SO LP formulation for DTA. Ziliaskopoulos' work has been received significant attention over the past few years. Li et al. (1999) expanded the LP formulation to allow for multi-origins and multi-destinations, although first in first out principles are not preserved with multiple destinations. Although there exist efficient algorithms for solving linear programs, such as the Simplex Algorithm (Dantzig, 1998), solving the SODTA on large transportation networks proved to be difficult as the size of the associated programs requires significant computational resources. In this respect, Li et al. (2003) proposed a Dantzig-Wolfe decomposition algorithm for the SODTA problem that reduces the computational intractability of the LP formulation. Additionally, the SODTA formulation has been expanded and applied to a variety of problems such as network design (Waller and Ziliaskopoulos, 2001; Ukkusuri and Waller, 2008) and signal control (Aziz and Ukkusuri, 2012), among many others.

In this paper, we consider the strategic traffic assignment problem, which seeks to represent some level of uncertainty in the travel decisions of transportation network users. The concept of strategic behaviour in travellers has been applied in a number of different settings. For example, Marcotte and Nguyen (1998) employ the concept in the adaptive sense, in which users know that an arc may be unavailable when they reach it during travel and therefore assign an ordered set of preferences to the succeeding arcs (Marcotte et al., 2004; Hamdouch et al., 2004). This work incorporates stochastic demand using the strategic approach as introduced by Waller et al. (2013). The strategy consists of assigning optimal path proportions that minimize expected travel time across multiple stochastic demand scenarios, not simply for an expected demand. This results in a two stage model, where optimal path proportions are identified a priori in the first stage (adaptive routing is not considered in this approach). In the second stage demand is realized, resulting in link volumes that vary between different demand scenarios. The benefits of this approach lie in its ability to predict varying link flows, although it is limited by the large size of the linear program. Additionally, Duell et al. (2013) extended the StrSODTA LP formulation to solve for the network design problem with promising results. This works builds on previous research on SODTA and aims to provide a novel solution method for the StrSODTA problem.

## 3 Problem Formulation

In this section we present the StrSODTA problem using the notation of Waller et al. (2013). In the traditional SODTA problem, the objective is to find the optimal time-path for each departure time from any node in the network. When a deterministic demand is loaded onto the network, that is, when the travel demand for every OD pair in the network is known, the optimal routes that minimize the TSTT can be found using the LP formulation proposed by Ziliaskopoulos (2000) for single destination problems, and the one of Li et al. (2003) for multiple OD problems. This LP formulation relies on the CTM to propagate the flow in the network. In contrast, strategic assignment aims to take into account the impact of demand uncertainty on traditional equilibrium planning models. In particular, strategic assignment assumes that the travel demand at a given departure time, for a given OD pair, is random variable with a known distribution. The StrSODTA formulation of Waller et al. (2013) aims to find the optimal proportion of flow on each path and for each departure time to minimize the expected TSTT across a range of discrete demand scenarios.

| Symbol | Description |
|---|---|
| $N$ | Set of Nodes |
| $A$ | Set of Arcs |
| $[c_{ij}]$ | Matrix of free-flow travel times for all $(i, j) \in A$ |
| $C$ | Set of cells |
| $C_r$ | Set of source cells |
| $C_s$ | Set of sink cells |
| $E$ | Set of connectors |
| $\Gamma^-(i)$ | Set of predecessor cells of cell $i$ |
| $\Gamma^+(i)$ | Set of successor cells of cell $i$ |
| $N_{t,i}$ | Maximum flow in cell $i$ at time interval $t$ |
| $Q_{t,i}$ | Maximum inflow and outflow of cell $i$ during time interval $t$ |
| $T$ | Set of time intervals |
| $T_D \subset T$ | Set of departure time intervals |
| $OD$ | Set of origin-destinations pairs: $\mu = (r, s) \in N^2$ |
| $D^{\xi,\mu}$ | Total Demand of OD pair $\mu$, in scenario $\xi$ |
| $D^{\xi,\mu}_\tau$ | Demand of OD pair $\mu$, at departure time $\tau$ in scenario $\xi$ |
| $\Xi$ | Set of demand scenarios |
| $p^\xi$ | Probability of demand scenario $\xi \in \Xi$ |
| $\delta^{\mu,\phi}_i$ | Parameter equal to 1 if cell $i \in C$ is in path $\phi \in \Phi(\mu)$, and 0 otherwise |
| $\delta^{\mu,\phi}_{ij}$ | Parameter equal to 1 if connector $(i, j) \in E$ is in path $\phi \in \Phi(\mu)$, and 0 otherwise |

Table 1: Mathematical Notation of Sets and Parameters

The mathematical notation used to represent the StrSODTA problem is summarized in Table 1. The network and the CTM related notation is identical to the one found in Ziliaskopoulos (2000) and in Waller et al. (2013). The strategic assignment related notation mainly consists of a set of finite demand scenarios $\xi \in \Xi$, each of which has a probability $p^\xi$ of occurr ing and we have:

$$\sum_{\xi \in \Xi} p^\xi = 1 \tag{1}$$

The travel demand for OD pair $\mu \in OD$ and departure time $\tau \in T_D$ in scenario $\xi$ is denoted $D^{\xi,\mu}_\tau$. The decision variables of the LP model for the StrSODTA problem are similar to the ones introduced by Ziliaskopoulos (2000) with the addition of the path proportion variable $\pi^{\mu,\phi}_\tau$. The decision variables are:

- $\pi^{\mu,\phi}_\tau \geq 0$ — Proportion of the demand which uses path $\phi \in \Phi(\mu)$ for OD $\mu$ at departure time $\tau \in T_D$.
- $x^{\xi,\mu,\phi}_{t,\tau,i} \geq 0$ — Flow in cell $i$ at departure time $\tau$, at time interval $t$, on path $\phi \in \Phi(\mu)$, OD $\mu$ in demand scenario $\xi$.
- $y^{\xi,\mu,\phi}_{t,\tau,ij} \geq 0$ — Flow from cell $i$ to cell $j$ at departure time $\tau$, at time interval $t$, on path $\phi \in \Phi(\mu)$, OD $\mu$ in demand scenario $\xi$.

Model 1 gives the LP formulation used in Waller et al. (2013) to represent the StrSODTA problem. The objective function (2) of the StrSODTA problem is to minimize the expected TSTT which is obtained by summing the flow variables of cells minus the sink cells, where the flow is routed. Similarly as in Ziliaskopoulos (2000), the CTM-decomposition of a network for SODTA assumes that both source and sink cells —which represent origin and destination nodes — have infinite capacity $N_{i,t}$ for all time intervals, and that the sink cells have infinite inflow $Q_{i,t}$ for all time intervals.

**Model 1** (Linear Program for StrSODTA)**.**

$$\min \sum_{\xi\in\Xi}\sum_{\mu\in OD}\sum_{\phi\in\Phi(\mu)}\sum_{t\in T}\sum_{\tau\in T_D}\sum_{i\in C\setminus C_s} p^{\xi} x^{\xi,\mu,\phi}_{t,\tau,i} \tag{2}$$

*subject to*

$$x^{\xi,\mu,\phi}_{t,\tau,i} - x^{\xi,\mu,\phi}_{t-1,\tau,i} - \sum_{j\in\Gamma^-(i)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t-1,\tau,ji} + \sum_{j\in\Gamma^+(i)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t-1,\tau,ij} = 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T^{\star}, \forall\tau\in T_D,\forall i\in C\setminus(C_r\cup C_s):\delta^{\mu,\phi}_{i}=1 \tag{3}$$

$$x^{\xi,\mu,\phi}_{t,\tau,i} - x^{\xi,\mu,\phi}_{t-1,\tau,i} - \sum_{i\in\Gamma^-(j)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t-1,\tau,ij} = 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T^{\star}, \forall\tau\in T_D,\forall i\in C_s:\delta^{\mu,\phi}_{i}=1 \tag{4}$$

$$x^{\xi,\mu,\phi}_{t,\tau,i} - x^{\xi,\mu,\phi}_{t-1,\tau,i} + \sum_{j\in\Gamma^+(i)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t-1,\tau,ij} = \pi^{\mu,\phi}_{\tau} D^{\xi,\mu}_{\tau} \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T^{\star}, \forall\tau\in OD,\forall i\in C_r:\delta^{\mu,\phi}_{i}=1 \tag{5}$$

$$\sum_{j\in\Gamma^+(i)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t,\tau,ij} - x^{\xi,\mu,\phi}_{t,\tau,i} \leq 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T, \forall\tau\in T_D,\forall i\in C\setminus C_s \tag{6}$$

$$\sum_{\mu\in OD}\sum_{\phi\in\Phi(\mu)}\sum_{\tau\in T_D}\left(\sum_{i\in\Gamma^-(j)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t,\tau,ij} + x^{\xi,\mu,\phi}_{t,\tau,i}\right) \leq N_{t,i} \quad \forall\xi\in\Xi,\forall t\in T,\forall i\in C\setminus(C_r\cup C_s) \tag{7}$$

$$\sum_{\mu\in OD}\sum_{\phi\in\Phi(\mu)}\sum_{\tau\in T_D}\sum_{i\in\Gamma^-(j)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t,\tau,ij} \leq Q_{t,i} \quad \forall\xi\in\Xi,\forall t\in T,\forall i\in C\setminus C_s \tag{8}$$

$$\sum_{\mu\in OD}\sum_{\phi\in\Phi(\mu)}\sum_{\tau\in T_D}\sum_{j\in\Gamma^+(i)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t,\tau,ij} \leq Q_{t,i} \quad \forall\xi\in\Xi,\forall t\in T,\forall i\in C\setminus C_r \tag{9}$$

$$\sum_{\phi\in\Phi(\mu)} \pi^{\mu,\phi}_{\tau} = 1 \quad \forall\mu\in OD,\forall\tau\in T_D \tag{10}$$

$$y^{\xi,\mu,\phi}_{0,\tau,ij} = 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall\tau\in T_D, \forall(i,j)\in E \tag{11}$$

$$y^{\xi,\mu,\phi}_{t,\tau,ij} \geq 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T, \forall\tau\in T_D,\forall(i,j)\in E \tag{12}$$

$$x^{\xi,\mu,\phi}_{t,\tau,i} \geq 0 \quad \forall\xi\in\Xi,\forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall t\in T, \forall\tau\in T_D,\forall i\in C \tag{13}$$

$$\pi^{\mu,\phi}_{\tau} \geq 0 \quad \forall\mu\in OD,\forall\phi\in\Phi(\mu),\forall\tau\in T_D \tag{14}$$

Constraints (3) is the cell mass conservation constraints which ensure that the flow is correctly propagated into the network. Constraints (4) and (5) ensure that the cell mass balance is correctly adjusted at the sink and source cells respectively. In particular, (5) loads the demand for each OD pair, each departure time and each scenario in the network through the path proportion variables. Constraint (6) ensure that the outflow of a cell is no more than its current occupancy. Constraints (7), (8) and (9) are the capacity constraints and ensure that the jam density of each cell $N_{i,t}$ is not violated, as well as the maximum inflow and outflow $Q_{i,t}$ between any two adjacent cells. Constraint (10) is the path proportion constraint and ensures that the traffic for a given OD pair and a departure time is distributed among the paths for this OD pair in the network. Finally, constraint (11) sets the initial flow in the connectors to the zero and (12), (13) and (14) are the non-negativity constraints.

While Model 1 can be solved using efficient algorithms for LP, the number of variables and constraints increase exponentially with the size of the network. This is due to the fact, that both decision variables and some constraints are indexed by the number paths between every OD pair. Since even in moderate size networks, the number of paths per OD may be sufficiently large to compromise the computational efficiency of this LP formulation, it is critical to develop adapted solution methods for the StrSODTA problem. In the following section, we present a novel solution algorithm for the StrSODTA problem.

# 4 Solution Algorithm for StrSODTA

We start by describing the path enumeration procedure used to limit the size of the input for Model 1. We then introduce a lower bound on the optimal expected TSTT that can be used to measure the gap to the optimal strategic assignment and present the pseudo-code of the solution algorithm for reducing the size of the input and solving the StrSODTA problem.

The LP formulation proposed for the StrSODTA seeks to find the optimal path proportion among for every OD pair and departure time in a network where stochastic demand scenarios are assumed. As we know from the vast literature on SO assignment problems, the optimal paths used to route the traffic from origins to destinations depend on the travel demand for this OD pair but may not require the usage of all paths between this OD pair. In fact, in large networks for a given travel demand, several paths may not be used because they turn out to be simply too costly. While it is not straightforward to identify those paths a priori, a simple heuristic is to enumerate paths by increasing length using the free-flow travel time on each link.

To provide an efficient method to reduce the size of the LP formulation for StrSODTA, we propose to bound the maximum number of paths per OD pair by an integer $k$ and determine the expected TSTT $z_k^\star$ for increasing values of $k$. Let $z^\star$ denote the optimal expected TSTT when all paths are available, by construction there exists a value $K$ such that $z^\star = z_K^\star$. The path enumeration procedure can be executed in polynomial time using a generic $k$ loopless shortest paths algorithm (Yen, 1971) and the free-flow travel time on links. This path enumeration procedure should be used until an optimal assignment is attained. However there is no direct procedure that can be used to prove the optimality of an assignment, hence we use a lower bound on the expected TSTT to measure the progress of the solutions.

**Model 2** (Relaxed Linear Program for StrSODTA)**.**

$$\min \sum_{\xi \in \Xi} \sum_{\mu \in OD} \sum_{t \in T} \sum_{\tau \in T_D} \sum_{i \in C \setminus C_s} p^{\xi} x^{\xi,\mu}_{t,\tau,i} \tag{15}$$

*subject to*

$$x^{\xi,\mu}_{t,\tau,i} - x^{\xi,\mu}_{t-1,\tau,i} - \sum_{j \in \Gamma^-(i)} y^{\xi,\mu}_{t-1,\tau,ji} + \sum_{j \in \Gamma^+(i)} y^{\xi,\mu}_{t-1,\tau,ij} = 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in T_D, \forall i \in C \setminus (C_r \cup C_s) \tag{16}$$

$$x^{\xi,\mu}_{t,\tau,i} - x^{\xi,\mu}_{t-1,\tau,i} - \sum_{i \in \Gamma^-(j)} y^{\xi,\mu}_{t-1,\tau,ij} = 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in T_D, \forall i \in C_s \tag{17}$$

$$x^{\xi,\mu}_{t,\tau,i} - x^{\xi,\mu}_{t-1,\tau,i} + \sum_{j \in \Gamma^+(i)} y^{\xi,\mu}_{t-1,\tau,ij} = D^{\xi,\mu}_{\tau} \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in OD, \forall i \in C_r \tag{18}$$

$$\sum_{j \in \Gamma^+(i)} y^{\xi,\mu}_{t,\tau,ij} - x^{\xi,\mu}_{t,\tau,i} \leq 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T, \forall \tau \in T_D, \forall i \in C \setminus C_s \tag{19}$$

$$\sum_{\mu \in OD} \sum_{\tau \in T_D} \left( \sum_{i \in \Gamma^-(j)} y^{\xi,\mu}_{t,\tau,ij} + x^{\xi,\mu}_{t,\tau,i} \right) \leq N_{t,i} \quad \forall \xi \in \Xi, \forall t \in T, \forall i \in C \setminus (C_r \cup C_s) \tag{20}$$

$$\sum_{\mu \in OD} \sum_{\tau \in T_D} \sum_{i \in \Gamma^-(j)} y^{\xi,\mu}_{t,\tau,ij} \leq Q_{t,i} \quad \forall \xi \in \Xi, \forall t \in T, \forall i \in C \setminus C_s \tag{21}$$

$$\sum_{\mu \in OD} \sum_{\tau \in T_D} \sum_{j \in \Gamma^+(i)} y^{\xi,\mu}_{t,\tau,ij} \leq Q_{t,i} \quad \forall \xi \in \Xi, \forall t \in T, \forall i \in C \setminus C_r \tag{22}$$

$$y^{\xi,\mu}_{0,\tau,ij} = 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall \tau \in T_D, \forall (i,j) \in E \tag{23}$$

$$y^{\xi,\mu}_{t,\tau,ij} \geq 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T, \forall \tau \in T_D, \forall (i,j) \in E \tag{24}$$

$$x^{\xi,\mu}_{t,\tau,i} \geq 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T, \forall \tau \in T_D, \forall i \in C \tag{25}$$

$$\tag{26}$$

The complexity of Model 1 arises from the large number of paths per OD pair which generates many variables and constraints. To find a lower bound on the optimal expected TSTT obtained after solving Model 1, we propose to solve a relaxed version of this model which does not incorporate path information. Model 2 summarizes this relaxation. In this LP formulation the path proportion variables are omitted and the flow variables have 5 indexes instead of 6. In contrast to Model 1, the size of Model 2 increases linearly with the size of the input.

**Proposition 1.** *Let $z^\star$ be the optimal expected TSTT obtained by solving Model 1 and let $z^\star_{LB}$ be the optimal expected TSTT obtained by solving Model 2; for any instance, we have:*

$$z^\star \geq z^\star_{LB} \tag{27}$$

*Proof.* To see that the optimum of Model 2 is a lower bound on the optimal expected TSTT obtained by solving Model 1, we show that the solution space defined by the latter LP formulation is tighter than the former. Let $\mu \in OD$ be an OD pair, we define the variables $\bar{x}^{\xi,\mu}_{t,\tau,i}$ and $\bar{y}^{\xi,\mu}_{t,\tau,ij}$ as follows:

$$\bar{x}^{\xi,\mu}_{t,\tau,i} = \sum_{\phi \in \Phi(\mu)} \delta^{\mu,\phi}_{i} x^{\xi,\mu,\phi}_{t,\tau,i}, \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T, \forall \tau \in T_D, \forall i \in C \tag{28}$$

$$\bar{y}^{\xi,\mu}_{t,\tau,ij} = \sum_{\phi \in \Phi(\mu)} \delta^{\mu,\phi}_{ij} y^{\xi,\mu,\phi}_{t,\tau,ij}, \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T, \forall \tau \in T_D, \forall (i,j) \in E \tag{29}$$

By summing the cell mass balance constraints (3), (4) and (5) over all paths $\phi \in \Phi(\mu)$, for each $\mu \in OD$, we obtain three constraints of the form:

$$\bar{x}^{\xi,\mu}_{t,\tau,i} - \bar{x}^{\xi,\mu}_{t-1,\tau,i} - \sum_{j \in \Gamma^-(i)} \bar{y}^{\xi,\mu}_{t-1,\tau,ji} + \sum_{j \in \Gamma^+(i)} \bar{y}^{\xi,\mu}_{t-1,\tau,ij} = 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in T_D, \forall i \in C \setminus (C_r \cup C_s) \tag{30}$$

$$\bar{x}^{\xi,\mu}_{t,\tau,i} - \bar{x}^{\xi,\mu}_{t-1,\tau,i} - \sum_{i \in \Gamma^-(j)} \bar{y}^{\xi,\mu}_{t-1,\tau,ij} = 0 \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in T_D, \forall i \in C_s \tag{31}$$

$$\bar{x}^{\xi,\mu}_{t,\tau,i} - \bar{x}^{\xi,\mu}_{t-1,\tau,i} + \sum_{j \in \Gamma^+(i)} \bar{y}^{\xi,\mu}_{t-1,\tau,ij} = D^{\xi,\mu}_{\tau} \sum_{\phi \in \Phi(\mu)} \pi^{\mu,\phi}_{\tau} \quad \forall \xi \in \Xi, \forall \mu \in OD, \forall t \in T^\star, \forall \tau \in OD, \forall i \in C_r \tag{32}$$

Since all paths per OD pair are considered in Model 1, we have $\bar{x}^{\xi,\mu}_{t,\tau,i} = x^{\xi,\mu}_{t,\tau,i}$ and $\bar{y}^{\xi,\mu}_{t,\tau,ij} = y^{\xi,\mu}_{t,\tau,ij}$. Furthermore, constraint (10) in Model 1 imposes that $\sum_{\phi \in \Phi(\mu)} \pi^{\mu,\phi}_{\tau} = 1$, hence the above three constraints are equivalent to the cell mass balance constraints in Model 2. The same process can be applied to show that the remaining constraints of Model 1 can be aggregated to form the constraints of Model 2. Since for each OD pair, the constraints of Model 2 represent linear combinations of the constraints of Model 1 over the set of paths between this OD pair, the convex hull defined by the LP of the former model contains the one defined by the latter model. □

Model 2 can be seen as a path-aggregated version of Model 1 and can be used to provide an estimation of the quality of a solution of the latter model. To measure the progress of the assignment, that is the evolution of $z^\star_k$ when $k$ increases, we define the relative optimality gap $r(k)$:

$$r(k) = \frac{z^\star_k - z^\star_{LB}}{z^\star_{LB}} \tag{33}$$

It should be clear that there is no guarantee that $r(k)$ eventually tends to zero when $k$ increases. Since the convex hull defined by Model 2 contains the one defined by Model 1, some solutions of the path-aggregated model may not be feasible for the original StrSODTA model. Consequently, it is necessary to monitor the evolution of $r(k)$ to decide when a satisfactory assignment has been reached. The solution algorithm used to solve Model 1 is summarized by Algorithm 1 where $\epsilon$ and $\epsilon'$ are tuning parameters used to decide when does the relative gap $r(k)$ is low enough or stable enough to stop the procedure.

# 5 Case Study

In this section, we implement Algorithm 1 on two well-studied transportation networks for traffic assignment, the Nguyen-Dupuis network and the Sioux-Falls network. For both case studies, we use the following parameters:

**Algorithm 1:** Path-enumeration Algorithm for StrSODTA

**Data**: Network $G = (N, A, [c_{ij}])$, Demand Scenarios for OD pairs
**Result**: Optimal path proportions for StrSODTA
Decompose $G$ in a cell network $C(G)$ using the CTM;
$z^\star_{LB} \leftarrow$ Solve Model 2;
$\forall \mu \in OD$, $\Phi(\mu) \leftarrow \emptyset$;
**while** $Convergence = false$ **do**
  **for** $\mu \in OD : \mu = (r, s) \in N^2$ **do**
    Add the $k$th shortest path from $r$ to $s$ in $G$ to $\Phi(\mu)$;
  **end**
  $z^\star_k \leftarrow$ Solve Model 1 with current paths sets;
  **if** $r_k = 0$ **then** Convergence $\leftarrow$ *true*;
  **else**
    Compare the value of $r(k)$ with previous values;
    **if** $r(k) \leq \epsilon$ *or* $r(k) = ... = r(k - \epsilon')$ **then** Convergence $\leftarrow$ *true*;
  **end**
  $k \leftarrow k + 1$;
**end**

- 3 demand scenarios $\xi \in \Xi$ with probabilities $p^\xi$ of 0.5, 0.3 and 0.2 respectively.
- A discretization time step for the CTM of 6 seconds.

The link free-flow travel-time used for both networks are from Bar-Gera (website accessed on January 2014) and cell capacities, inflow and outflow are derived using the discretization time step. The CTM decomposition and the generation of the $k$ shortest paths are performed using C++ routines. The AMPL/CPLEX framework is used to coordinate the resolution of the linear programs.

## 5.1 Nguyen-Dupuis

For the Nguyen-Dupuis case study, we use the directed network depicted by Fig. 1 where nodes 1 and 4 are used as origins and nodes 2 and 3 as destinations. Due to the unilateral nature of the network, only a limited number of paths per OD pair exist. In this case study, we simulate a total of 8 departures times and we consider two distributions of the total travel demand across the possible departures times:

1. Uniform — the total demand is uniformly distributed across the departure times, hence for each scenario $\xi$ and OD $\mu$, $D^{\xi,\mu}_\tau = D^{\xi,\mu}/8$.
2. Normal — the total demand is normally distributed across the departure times with a distribution of mean $D^{\xi,\mu}$ and a unit variance.

We examine two traffic configurations: *pairwise*, where the travel demand exists only among some OD pairs and *mixed*, where the travel demand is loaded across every combination of origin and destination. Table 2 summarizes the values used for all traffic and demand scenarios. The results obtained are presented in Fig. 2.

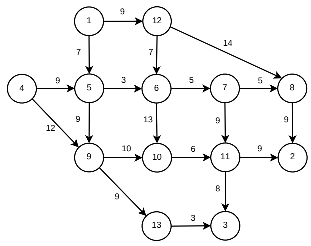



Figure 1: Nguyen-Dupuis Network

| $\mu$ \ $\xi$ | Pairwise 1 | Pairwise 2 | Pairwise 3 | Mixed 1 | Mixed 2 | Mixed 3 |
|---|---|---|---|---|---|---|
| (1,2) | 10 | 11 | 12 | 10 | 11 | 12 |
| (1,3) | 0 | 0 | 0 | 10 | 11 | 12 |
| (4,2) | 0 | 0 | 0 | 10 | 11 | 12 |
| (4,3) | 10 | 11 | 12 | 10 | 11 | 12 |

Table 2: Total Travel Demand for Pairwise and Mixed Traffic

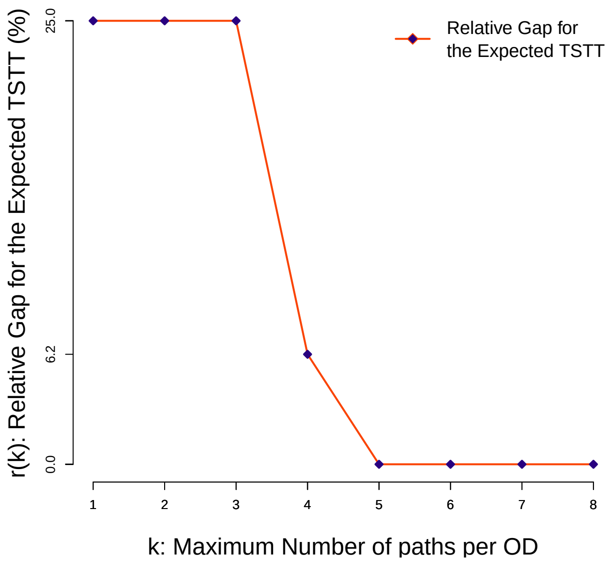
Fig. 2a shows the results obtained with a uniform travel demand across the departure times and a pairwise traffic. We report that the relative gap $r(k)$ remains unchanged for $k = 1, 2, 3$ with $r(k) = 23.3\%$. Using an additional path per OD, that is $k = 4$, $r(k)$ drops to 7.1% and the gap to optimality is closed for $k \geq 5$. It should be noted that in this network the total number of paths among the OD pairs considered, *i.e.* from node 1 to node 2 and from node 4 to node 3, is 8; hence all possible paths are eventually generated. In this case study, the lower bound on the expected TSTT is binding and the results for $k = 6, 7, 8$ are shown for description purposes only. Fig. 2c shows the results obtained with a normal distribution for the same pairwise traffic. In this scenario, the relative gap is almost equal to zero for $k \geq 5$.

Fig. 2b and Fig. 2d show the performance of the solution algorithm when a mixed traffic is used. We observe that although the trend of $r(k)$ is similar to the pairwise case study, the optimality gap is not as much reduced. This is potentially a consequence of the relaxation used to derive the lower bound on the optimal expected TSTT: since Model 2 aggregates the flow variables across the paths of the network, the tightness of the lower bound is decreased whenever the paths used by the traffic among different OD pairs share some common links. In the uniform and pairwise traffic scenario, the demand between both OD pairs is sufficiently low such that the set of paths used for an OD pair is not used for the other OD pair; therefore the lower bound is able to match the optimal expected TSTT.

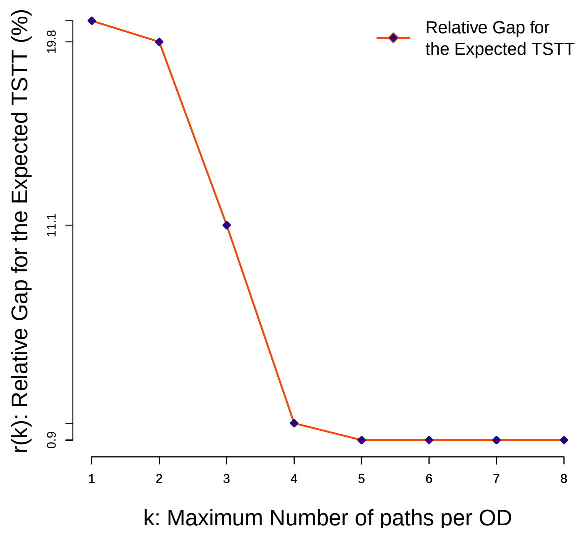
## 5.2 Sioux-Falls

For the Sioux-Falls case study, we use the network depicted by Fig. 3 with 4 origins and 4 destinations nodes and a uniform demand distributed across two departures times. Table 3 gives the total demand used per OD pair in scenario 1 and this total demand is inflated by 10% and 20% in demand scenarios 2 and 3 respectively. The results obtained are presented in Fig. 4.

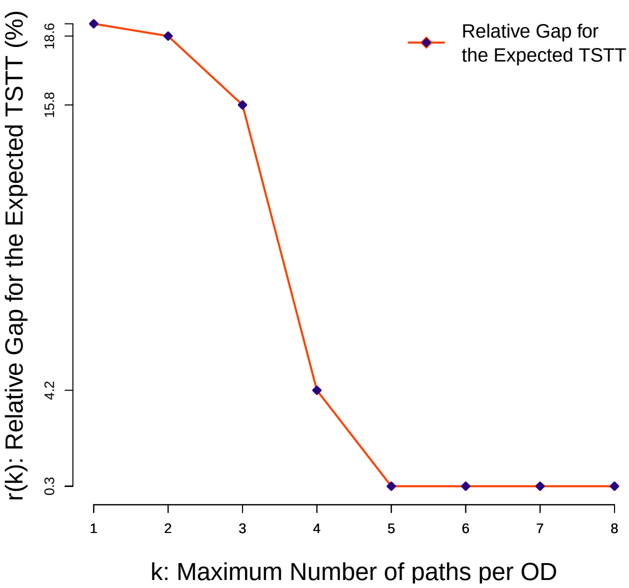
Fig. 4a shows that the relative gap is initially, that is for a single path per OD pair, of 43.1 %. This figure is progressively reduced to 12.3% for $k = 13$ and appears to be stabilized at 11.8% for $k \geq 15$. The algorithm is stopped for $k = 20$ after observing that the relative gap remains unchanged for five consecutive iterations. It should be noted that the detection of flats segment is not a sufficient condition to ensure the optimality of the traffic assignment, as the order in which the paths are enumerated does not guarantee that their utility will decrease. As depicted by Fig. 4b, the total number of paths increases linearly with $k$, which suggests that not all paths per OD pair have yet been used in the LP formulation. The size of the instance significantly affects the LP solve time and as expected it is strongly correlated with the maximum number of paths per OD.



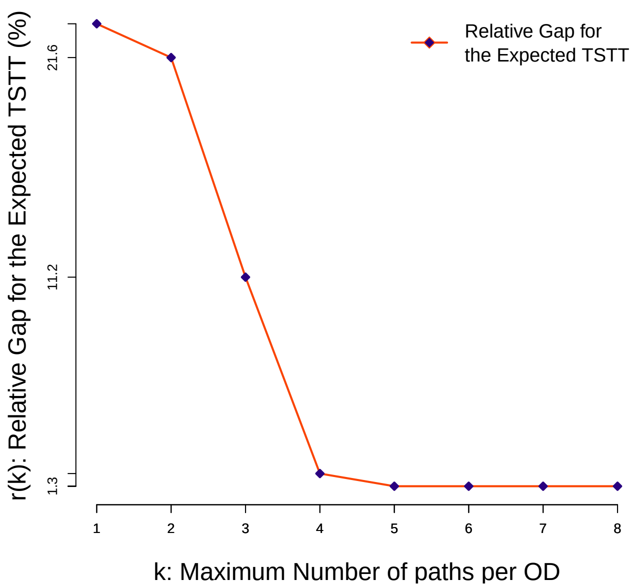

(a) Uniform & Pairwise




(b) Uniform & Mixed




(c) Normal & Pairwise




(d) Normal & Mixed

Figure 2: Results for the Nguyen-Dupuis case study

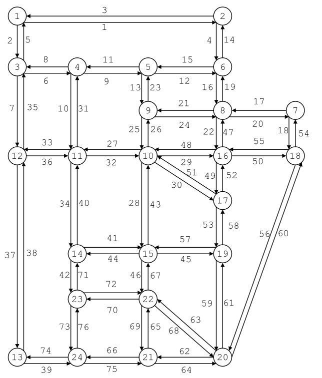


Figure 3: Sioux-Falls

| OD pairs | 1 | 2 | 3 | 13 |
|---|---|---|---|---|
| 20 | 0 | 31 | 42 | 20 |
| 18 | 28 | 0 | 16 | 23 |
| 7 | 33 | 25 | 0 | 27 |
| 6 | 20 | 26 | 19 | 0 |

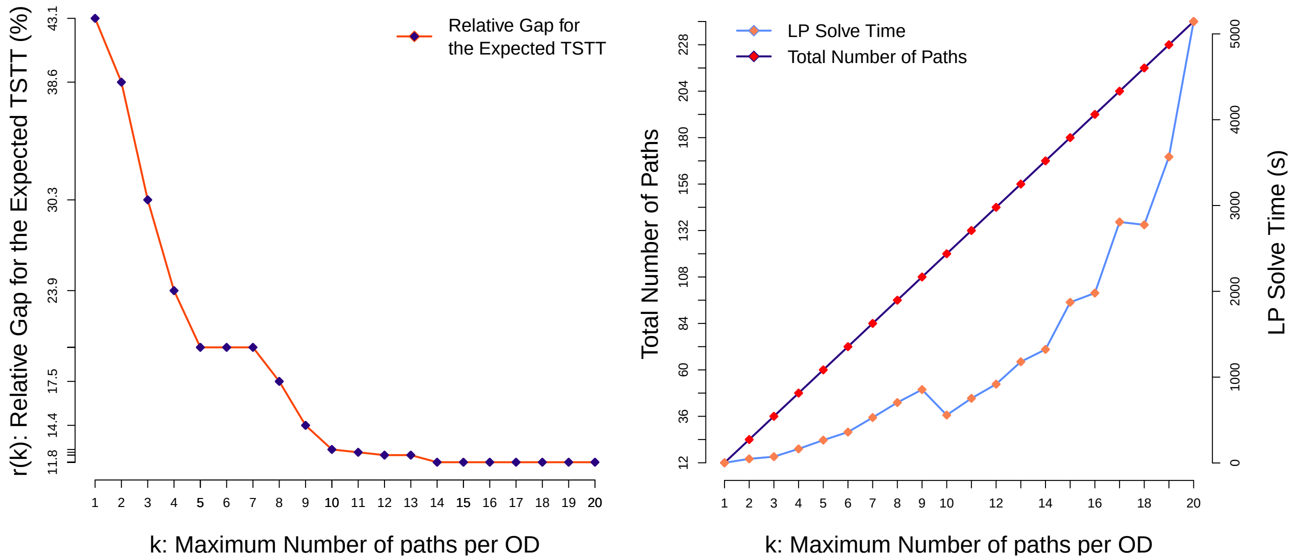
Table 3: Travel Demand

Although a more satisfactory convergence criterion for Algorithm 1 remains to be found, the solution algorithm proposed for the StrSODTA problem is able to provide good solutions in a reasonable amount of time. Furthermore, the lower bound on the optimal expected TSTT provides a metric to measure the quality of the traffic assignments obtained at every iteration of the solution algorithm which represents valuable insight for researchers and practitioners.

A possible way to determine whether an assignment found by the solution algorithm is optimal is to investigate the role of the dual variables of Model 1. In the context of single destination SODTA, Ziliaskopoulos (2000) showed that the dual variables of the cell mass balance constraints of the deterministic SODTA LP formulation represent the time-dependent least marginal cost path from a given cell, at a given time, to the destination node. Li et al. (2003) have shown that this LP formulation could be extended to the multiple OD case. Therefore, it is potentially of interest to study the implications of the dual variables of the cell mass balance constraints in the context of the StrSODTA problem. In particular, if one is able to determine the expected marginal cost of a path for a given OD pair, at a given departure time and time interval; this information could be used to evaluate the optimality of a strategic traffic assignment. However, this approach may require to be able to compute the cost of the least expected time-dependent marginal cost path, which is known to be a NP-hard problem (Miller-Hooks and Mahmassani, 2000).

## 6 Conclusion

In this paper, we have introduce a novel solution algorithm to solve the StrSODTA problem. Although the StrSODTA problem can be addressed using a LP formulation, the size of the model grows exponentially with the size of the input. This a consequence of the path-based formulation which is used in the strategic assignment paradigm. Consequently, a computationally efficient approach is required to solve this LP on large transportation networks. We have proposed a solution algorithm which relies on a path enumeration technique to progressively add paths to the StrSODTA model. Namely we use a $k$ loopless shortest path algorithm to sequentially enumerate paths among the OD pairs of the network. The $k$ shortest path algorithm is executed using free-flow travel time as link cost and thus does not guarantee that the paths are enumerated in the correct order with regards to path utility at SO conditions. In order to measure the progress of the solutions provided by the solution algorithm, we have introduced a novel lower bound on the optimal expected TSTT which is obtained by solving a relaxed version of the original StrSODTA




(a) Evolution of the relative gap

(b) Evolution of the total number of paths and the LP solve time

Figure 4: Results for the Sioux-Falls case study

model. We have reported results on two case studies using benchmark transportation networks for traffic assignment problems. Our results show that the proposed solution algorithm is able to efficiently solve the StrSODTA within acceptable gap to optimality. Furthermore, we report that solving the complete, *i.e.* with all paths per OD, StrSODTA model may indeed require intensive computational resources, even on moderate size networks.


## Acknowledgments

NICTA is funded by the Australian Department of Communications and the Australian Research Council through the ICT Centre of Excellence program.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