STRATEGIC SYSTEM RELIABLE FORMULATION FOR TRAFFIC ASSIGNMENT

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ABSTRACT

Traditionally, planning and evaluation of transport systems has focused on the idea that minimizing travel time will result in a better system for its users. However, planners are increasingly recognizing the value users place on a reliable system, i.e., a system that experiences less fluctuation in travel times on individual links. This has challenged researchers to develop tools to evaluate reliability in transportation networks. This research develops a novel approach to evaluate performance and reliability in transportation networks by proposing a system reliable formulation that minimizes the system variance in optimal conditions. This model incorporates strategic traffic assignment in system optimal conditions, where users are routed to account for demand uncertainty. One application of this concept is the pricing of networks to create a more reliable system for users. Results for the system reliable formulation are demonstrated and compared to the results from a strategic system optimal routing.

1. INTRODUCTION

Recent studies have found that drivers value travel time reliability somewhere between 50 and 80 percent of their value of travel time (Asensio and Matas, 2008; Hollander, 2006; Zheng et al., 2010). These findings have challenged planners to develop methods to develop a reliable transportation networks. This work presents a novel formulation for Strategic System Reliable (StrSR) traffic assignment that will help planners determine optimal routing for users to minimize system variance and thus creating a reliable transportation network.

Transportation models must address a number of uncertainties to quantify reliability. In particular, uncertainty results from variability associated with the demand that is an input for traffic assignment models. The traditional system optimal (SO) formulation is a relatively simple problem in which optimal routes are determined that will minimize the total system travel time (Wardrop, 1952). The SO model provides guidance towards developing transportation management strategies that reduce the total system travel time and other important system characteristics, although it is not a reflection of actual user behaviour or choices. Additionally, this model is important for planners to determine optimal routing strategies, particularly in regards to determining optimal marginal social cost-based tolls, and may become increasingly important with the advent of autonomous vehicles. However, because this formulation is static and based on deterministic demands, it is not possible to evaluate reliability. This work addresses this challenge by incorporating strategic traffic assignment proposed by Dixit et al. (2013).

Specifically, this paper presents a detailed derivation and formulation for StrSR assignment problem and compares the results with the flows from a traditional strategic SO (StrSO) network. This research develops a novel framework to evaluate network performance and reliability in a transportation network. Using the strategic approach, this work quantifies the link level variance of the system, and then uses this analytical measurement as an objective to determine vehicle routing patterns. This results in a system reliable formulation as a convex programming problem that minimizes the system variance in optimal conditions. A brief discussion of background and motivation for this problem is provided in Section 2, while Section 3 contains the detailed derivation and formulation of the strategic system optimal (StrSO) traffic assignment and StrSR traffic assignment formulations. Section 4 discusses the results of these models on an example network, and Section 5 concludes this paper with potential directions for future work.

2. BACKGROUND AND MOTIVATION

As noted, this paper presents a new formulation for traffic assignment that captures an aspect of reliability in the transport system, and accounts for travel demand uncertainty. In order to find the network performance with the least variability, it is based on a system optimal approach to determining route flows, the formulation for which is explained in Sheffi (1985). Optimal route assignment is not based on user behaviour, or the equilibrium concept of users choosing the shortest path. In that way, SO is not a model used to represent observed vehicle flow patterns, although it is worth noting that the discussion of user equilibrium (UE) versus SO behaviour and its interpretation in observed networks dates back many years (Boyce,1979; LeBlanc and Abdulaal, 1984).

Traffic assignment models based on optimality play an important role in the transport planning process. One application of system optimal flows is used to determine the optimal marginal social cost-based tolls in a transportation network (Gardner et al, 2008; Gardner et al, 2010). A system reliable formulation could potentially be used in a similar way to create more reliable network for users. To the best of the authors knowledge, this paper is the first to propose the concept of a system reliable formulation.

Accounting for uncertainty in transport planning encompasses network analysis under a variety of possible scenarios rather than the unrealistic assumption of deterministic conditions, and is becoming an increasingly important focus in the area of transport modelling. Uncertainty arises from a number of sources, including travel demand uncertainty (Bell et al, 1999), supply and capacity uncertainty (Chen et al, 1999; Lo and Tung, 2003), and uncertainty associated with the behavioural decisions of network users (Liu et al, 2002). In particular, this research addresses long-term uncertainty in demand that results in a day-to-day volatility that is not the result of recurrent events (Yin and Ieda, 2001). It is important for transport planners to explain this source of uncertainty because as Waller et al (2001) show, neglecting the impact of long term demand uncertainty by using a single fixed estimate of future demand can result in significant underestimation of the future system performance.

Strategic dynamic traffic assignment was presented by Fajardo et al (2012) and formulated using a linear programming approach. This is the first work to use strategic in the sense of users developing path strategies to minimize travel time over the range of a travel demand distribution. Fajardo et al present a dynamic formulation for the strategic system optimal traffic assignment problem and demonstrate a solution method using an example network. Strategic user equilibrium (StrUE) introduced by Dixit et al (2013) further refines the strategic concept by assuming that people choose the shortest expected cost based on a demand distribution. Dixit et al demonstrate the capability of the strategic approach to capture link travel time by comparing the proposed analytical approach with a simulated approach on a realistic sized network.

Additionally, strategic traffic assignment models account for demand uncertainty by assigning route choice based on the proportion of flow on each link. Strategic equilibrium is reached when the expected travel costs are equal on all used paths, and is less than the cost on any unused path. The critical difference from traditional approaches is identifying path proportions/strategies for a given objective function and demand distribution.

In the strategic SO traffic assignment model, the optimal path proportions are based on a demand distribution so as to minimize the *expected* total system travel time. To the authors' knowledge, this is the first work to present a formulation for the Strategic System Reliable traffic assignment problem.

3. METHODOLOGY AND FORMULATION

This section further defines the concept of strategic system reliable traffic assignment, and then derives the formulation for both StrSO and StrSR. A list of the notation used in this section can be seen in Table 1.

This work incorporates the strategic approach to traffic assignment, a detailed formulation of which can be found in Dixit et al (2013). Unlike traditional traffic assignment formulations, strategic assignment assumes that the demand is a random variable with a known mean and standard deviation, and the flow on each link is a proportion of the aggregate demand that will remain the same for all realizations of demand. The physical interpretation of this is flows that appear to be a disequilibrium for any given manifestation (for example, a daily demand) but are in fact defined by a higher level optimization (or equilibrium) based on the demand distribution. An important assumption in this formulation is that the fractions for each OD pair rs is constant and is independent to the distribution of the total demand. Network-wide disruptions such as inclement weather support this assumption.

To support the formulation for strategic system reliable flow assignment we must first formulate a StrSO assignment. In StrSR, the flow patterns are optimized to account for the objective of reliability, as quantified by the variance of the total system travel time. StrSO determines the flow patterns to minimize the expected total system travel time and is analogous to the system optimal formulation for deterministic user equilibrium (Sheffi, 1984). Determining flow patterns to minimize system variability does not have an analog in traditional traffic assignment.

Table 1. Summary of notation for the strategic system optimal traffic assignment model

N	Node (index) set
A	Link (index) set
K_{RS}	Path set
p_a	Proportion of total demand on arc a; **f**=(…,p_a,…)
t_a	Travel time on arc a; **t**= (…,t_a,…)
f_k^{rs}	Proportion of flow on path k, connecting OD pair r-s; $\mathbf{f^{rs}}$= (…,f_k^{rs}, …); **f**=(…,f^{rs},…)
c_k^{rs}	Travel time on path k connecting O-D pair r-s; $\mathbf{c^{rs}}$= (…,c_k^{rs}, …); **c**=(…,c^{rs},…)
q_{rs}	Fraction of total trips that are between OD pair r-s; $1 = \sum_{\forall rs} q_{rs}$
T	Random variable for total trips with probability distribution $g(T)$
g_{rs}	Probability distribution for trip rates between origin r and destination s; $g(q) = \prod_{\forall rs} g(q_{rs})$
$\delta_{a,k}^{rs}$	Indicator variable $\delta_{a,k}^{rs} = \begin{cases} 1 \; if \; a \; is \; included \; in \; path \; k \\ 0 \; otherwise \end{cases}$ $(\Delta^{rs})_{a,k} = \delta_{a,k}^{rs}$; $\Delta = (\dots, \Delta^{rs}, \dots)$

In the system reliable routing paradigm users are directed to a strategy of routes determined by path proportions so as to minimize the variance in the total system travel time from all origins to destinations. We define path proportions f_i on path k, where path k belongs to the set K_{RS}, as the percent of OD demand between origin R and destination S that will travel on route i.

$$f = \{f_i\}_{i \epsilon K_{RS}} = \left\{ f \colon \sum_{i \epsilon K_{RS}} f_i = 1, \forall f_i \geq 0 \right\} \qquad [1]$$

The proportion of flow on a link is the sum of the path proportions that are incident on link a contained within a set of links A, and are described below.

$$p = \{ p_a \; \forall a \in A \colon p_a = \sum_{i \in K_{RS}} \delta_a^i \xi_i \, , \sum_{i \in K_{RS}} \xi_i = 1 \; \forall R, S \} \qquad [2]$$

3.1 Strategic system optimal traffic assignment

In StrSO, the flows will be distributed so as to minimize the sum of the total *expected* flow on each link, $\tilde{z}(p)$, where the flow is represented as the proportion of the total demand on each link [2]. The expected total system travel time is the sum of the travel time on each link, where the flow on a link is the proportion of the demand on that link p_a multiplied by the aggregate number of trips, which is a random variable T.

$$E[\tilde{z}(p)] = E[\sum_a p_a T \times t_a(p_a T)] = \sum_a E(p_a T \times t_a(p_a T)) \quad [3]$$

The expected system travel time for a general link cost function $t_a(p_a T)$ is then the integral of the flow on the link multiplied by the cost function of that link multiplied by the probability distribution function of the trip demand *g(T)*.

$$\sum_a E[p_a T t_a(p_a T)] = \sum_a \int_{-\infty}^{\infty} p_a T t_a(p_a T) g(T) dT \quad [4]$$

In order to provide the StrSO traffic assignment approach with a closed form analytical solution, we assume that the travel time on a link can be described by the well-known Bureau of Public Roads function:

$$t_a(p_a T) = \ t_f(1 + \alpha \left(\frac{p_a T}{C_a}\right)^{\beta}) \quad [5]$$

Where t_a is the travel time on link a, t_f is the free flow travel time on link a, C_a is the capacity of link a, α and β are shaping parameters based on link geometry, p_a is the proportion of the aggregate demand on link a, and T is a random variable representing the aggregate network demand with probability distribution $g(T)$. The BPR function is used to derive the expected total system travel time for the StrSO formulation.

$$E[\tilde{z}(p)] = \sum_a \int_{-\infty}^{\infty} \left(p_a t_f T + \frac{t_f \alpha}{c^{\beta}} (p_a T)^{\beta}\right) g(T) dT = \sum_a (t_f p_a M_1 + t_f \alpha \frac{p_a^{\beta+1}}{c^{\beta}} M_{\beta+1}) \quad [6]$$

Where M_k is the kth moment of the aggregate demand distribution $g(T)$.

Thus the mathematical programming function to solve for the StrSO flow distribution will be convex , can be solved using optimisation techniques.

$$\tilde{z}(p) = \sum_{a \in A} (t_f p_a M_1 + t_f \alpha \frac{p_a^{\beta+1}}{c^{\beta}} M_{\beta+1}) \quad [7]$$

$$\sum_k f_k^{rs} = q_{rs} \qquad \forall r, s \quad [8]$$

$$p_k^{rs} \geq 0 \qquad \forall k, r, s \quad [9]$$

$$p_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \qquad \forall a \quad [10]$$

Note the the constraints [8]-[10] are the same as from the strategic equilibrium program. Equation [8] loads the demand into the network, while equation [10] provides network structure by connecting link properties and path proportions.

3.2 Strategic system reliable traffic assignment

We define system reliability as the variance of the total system travel time. This captures the day-to-day fluctuations in the total system performance. Therefore system reliability using a general link cost function $t_a(p_aT)$ can be represented as:

$$Var\left(\sum_a p_aTt_a(p_aT)\right) = E\left[\left(\sum_a p_aTt_a(p_aT)\right)^2\right] - \left(E\left[\sum_a p_aTt_a(p_aT)\right]\right)^2 \qquad [11]$$

We assume that the link cost functions are independent of one another, and therefore their covariances are equal to zero. This means that the variance of the sum is equal to the sum of the variances. Using this property we can derive the system reliable objective function.

$$Var[\tilde{z}(p)] = E\left(\left(\sum_a p_at_fT + \frac{t_f\alpha}{c^\beta}(p_aT)^{\beta+1}\right)^2\right) - (E(\sum_a p_at_fT + \frac{t_f\alpha}{c^\beta}(p_aT)^{\beta+1}))^2 \qquad [12]$$

$$Var[\tilde{z}(p)] = \sum_a \Bigg(t_f^2p_a^2M_2 + 2\frac{t_f^2\alpha}{c^k}p_a^{\beta+2}M_{\beta+2} + \frac{t_f^2\alpha^2}{c^{2\beta}}p_a^{2\beta+2}M_{2\beta+2} - t_f^2p_a^2M_1^2 - 2\frac{t_f^2\alpha}{c^\beta}p_a^{\beta+2}M_1M_{\beta+1} - \frac{t_f^2\alpha^2}{c^{2k}}p_a^{2k+2}M_{k+1}^2\Bigg) \qquad [13]$$

The system variance $\bar{\bar{z}}(p)$ can then be formulated as the following program, with the same constraints as the StrSO program (equations [8]-[10]).

$$\bar{\bar{z}}(p) = \sum_{a\in A}\left[p_a^2t_{af}^2(M_2 - M_1^2) + 2\frac{\alpha t_{af}^2}{C_a^\beta}p_a^{\beta+2}(M_{\beta+2} - M_{\beta+1}M_1) + \frac{\alpha^2t_{af}^2}{C_a^{2\beta}}p_a^{2\beta+2}(M_{2+2\beta} - M_{\beta+1}^2)\right] \qquad [14]$$

$$\sum_k p_k^{rs} = q_{rs} \qquad \forall r,s \qquad [15]$$

$$p_k^{rs} \geq 0 \qquad \forall k,r,s \qquad [16]$$

$$f_a = \sum_r\sum_s\sum_k p_k^{rs}\delta_{a,k}^{rs} \qquad \forall a \qquad [17]$$

The variance in the system is a function of the higher moments of the probability distribution of the demand. For most distributions, the higher moments have no closed form analytical solutions. However, by assuming that the travel demand can be described using a lognormal distribution, this formulation can be solved. While not as intuitive as the deterministic SO or the

StrSO formulation, this objective remains convex and therefore can be solved using standard nonlinear programming optimization techniques. The results are demonstrated in the next section.

4. DEMONSTRATION OF RESULTS

The strategic system optimal flow patterns represent an optimal solution to the traffic assignment problem that accounts for the inherent uncertainty in travel demand. This formulation assumes that the travel time on a link can be described by the BPR cost function, that link travel times are independent, and that the proportion of total demand for each OD pair will remain the same for all demand realizations. This section demonstrates the results for the StrSO and StrSR mathematical programs that were derived in Section 3. These convex, nonlinear programs were solved using GAMS, a high-level modelling system for mathematical programming and optimization that is built for complex, large-scale modelling applications (Rosenthal, 2012).

The results for the StrSO and the StrSR flow models are demonstrated on the example network seen in Figure 1. This network consists of 6 nodes and 9 links with a capacity of 50 and BPR parameters of 0.15 and 4. There is one origin-destination pair between (1,6) with a lognormally distributed demand with a mean of 100 and a standard deviation of 20. The moment of a lognormal distribution can be analytically calculated as follows:

$$M_s = \exp\,(s\mu + 0.5s^2\sigma^2) \qquad [18]$$

Recall that the StrSO objective function represents the total system travel time, while the StrSR objective function is the variance of the total system travel time. There are eight paths in this network between the OD pair (1,6), and strategic approach will optimize the proportions of the demand on each path according to the lognormal demand distribution with known mean and variance.

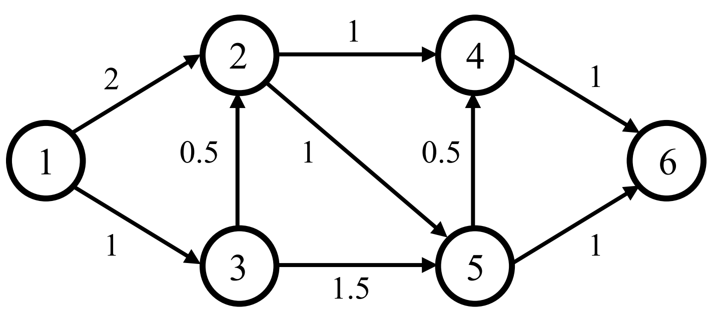

Figure 1. Test network with link lengths and node indexes listed

A network that is optimized for system variance versus system travel time that explicitly accounts for demand uncertainty will result in a more reliable network, where reliability is defined as less fluctuation between system travel times over the range of the demand distribution. While traditional optimal flow patterns are not a reflection of user behaviour, it does provide lower bound on the strategic equilibrium assignment problem and it is an important reflection of system characteristics.

The total system travel time (TSTT) and the variance of the TSTT for the case of optimizing for travel time versus reliability is displayed in Table 2. As expected, the TSTT is higher in the case of minimizing travel time variance. For the sake of comparison, the strategic equilibrium flows for this network results in a TSTT of 429.5 minutes. This relatively small difference between equilibrium and optimal flows is also seen in the deterministic case. The ~3% difference between the STD of each system is significantly greater than the nearly negligible difference between travel times.

Table 2. Objective function values for StrSO and StrSR

	StrSO Objective	StrSR Objective
TSTT (minutes)	424.48	427.7
System Variance	1418.13	1333.16
System STD (minutes)	37.66	36.51

Tables 3 and 4 contain the detailed results from the StrSO network and the StrSR network respectively, including the actual travel time on each link calculated using the BPR function [5], the link values for the StrSO and StrSR objectives [7] and [14], and finally the link variance. The StrSO objective is the aggregate expected travel time on that link, while the StrSR objective value is the variance of the aggregate travel time on that link. Note that although there are links with a variance of 0 in both cases, these links hold a different proportion of the flow in each system.

Table 3. Link results for the StrSO assignment

Strategic System Optimal Flows					
Link	Proportion of total demand on link	Travel time	SO Objective Value of Link	SR Objective Value of Link	Link Variance
(1,2)	0.43	2.17	96	368.08	0.01
(1,3)	0.57	1.25	73.44	445.78	0.03
(2,4)	0.45	1.1	49.89	105.78	0
(2,5)	0.27	1.01	27.9	17.8	0
(3,2)	0.29	0.51	14.62	5	0
(3,5)	0.28	1.52	42.55	41.73	0
(4,6)	0.47	1.11	53.02	132.24	0.01
(5,4)	0.02	0.5	0.96	0.02	0

(5,6)	0.53	1.2	66.11	301.71	0.02

Table 4. Link results for the StrSR assignment

Strategic System Reliable Flows					
Link	Proportion of total demand	Travel time	SO Objective Value of Link	SR Objective Value of Link	Link Variance
(1,2)	0.45	2.19	99.63	420.71	0.02
(1,3)	0.55	1.23	70.59	385.23	0.02
(2,4)	0.36	1.04	37.51	40.33	0
(2,5)	0.35	1.04	37.12	39.05	0
(3,2)	0.27	0.51	13.52	4.12	0
(3,5)	0.29	1.52	43.95	45.32	0
(4,6)	0.5	1.15	59.1	198.13	0.01
(5,4)	0.14	0.5	7.1	1.02	0
(5,6)	0.5	1.15	59.19	199.27	0.01

The path proportions that result from the traffic assignment are another an important consideration of the strategic approach. These are the proportions that will represent an optimal system considering all demand realizations. A comparison of these proportions for the StrSO and the StrSR systems can be seen in Table 5. Although these proportions change in the determined results of the two systems, thus validating the model formulation as presented in Section 3, it is still a challenge to interpret what this change may imply in the system.

Table 5. A comparison between the path proportions on the example network for StrSO and StrSR flows

Path	Links	StrSO	StrSR
1	1-3-7	**0.21**	**0.12**
2	1-4-9	**0.23**	**0.26**
3	1-4-8-7	**0**	**0.06**
4	2-6-9	**0.26**	**0.21**
5	2-6-8-7	**0.02**	**0.08**
6	2-5-3-7	**0.24**	**0.24**
7	2-5-4-9	**0.05**	**0.03**
8	2-5-4-8-7	**0**	**0**

Finally, it is important to note that the proportion of the flow one each link changes according to the model formulation. Table 6 presents the results on the example network for the proportion of the total demand on each link for the StrSO, StrSR, deterministic SO, and StrUE models. The StrSO and deterministic proportions are very similar, although not identical. All links show non-negligible differences between the results from the different models.

Table 6. Comparison of the link flow proportions between StrSO, StrSR, deterministic SO, and StrUE

Link	StrSO	StrSR	Deterministic SO	StrUE
	Proportion of flow on each link			
(1,2)	0.43	0.45	0.43	0.33
(1,3)	0.57	0.55	0.57	0.67
(2,4)	0.45	0.36	0.46	0.46
(2,5)	0.27	0.35	0.26	0.23
(3,2)	0.29	0.27	0.29	0.36
(3,5)	0.28	0.29	0.28	0.31
(4,6)	0.47	0.50	0.46	0.46
(5,4)	0.02	0.14	0.00	0.00
(5,6)	0.53	0.50	0.54	0.54

5. CONCLUSION

This work has introduced the novel concept of strategic system reliable traffic assignment, where the optimal flow pattern is found based on minimizing the variance of the total system travel time. This concept is possible because of the characteristics of the strategic approach, where flows are determined to optimize *expected* travel time on all links and the demand is a random variable with a known mean and variance. This work specifically assumes that the link costs can be described by the BPR function and that the demand is lognormally distributed, both of which are well grounded assumptions in the research literature. In summary, this work argued:

- Users place a high value on reliability as well as reducing travel time;
- The strategic approach allows transport network planners to quantify the link level variance based on the distribution of the travel demand;
- A system reliable approach can be used to minimize this variance.

While the System Reliable approach to traffic assignment is still in its infancy, it could allow planners to gleam valuable information about predicting or even designing for a more reliable system.

Works Cited

Asensio, J. and Matas, A. (2008) Commuters' valuation of travel time variability, *Transportation Research Part E*, 44 1074–1085.

Boyce, D. E. (1979), Introduction to the special issue on network design methodologies, *Transportation Research*, **13B**, 1–3.

Dixit, V., L. Gardner, S. T. Waller (2013), Strategic User Equilibrium Assignment under Trip Variability, Accepted for Presentation at the 92nd Annual Meeting of the Transportation Research Board.

Fajardo, D., Waller, S.T., Duell, M. & Dixit, V. (2012). Linear programming formulation for strategic dynamic traffic assignment. Proceedings of the 4th International Symposium on Dynamic Traffic Assignment.

Gardner, L., S. Boyles, S.T. Waller, (2011). Quantifying the Benefit of Responsive Pricing and Travel Information in the Stochastic Congestion Pricing Problem. Transport. Res. Part A, 45(3), pp. 204-218.

Gardner, L., A. Unnikrishnan and S.T. Waller (2008). "Robust Pricing for Transportation Networks under Uncertain Demand". Journal of the Transportation Research Board, No. 2085, 21-30. 2008.

LeBlanc, L. J. & Abdulaal, M. (1984), A comparison of the user-optimum versus system-optimum traffic assignment in transportation network design, Transportation Research, 18B, 115–21.

Liu, H.X., Ban, X., Ran, B., Mirchandani, P.,, (2002). An analytical dynamic traffic assignment model with probabilistic travel times and perceptions. In: Paper Presented at 81st annual Transportation Research Board Meeting, Washington, DC.

Rosenthal, R (2012). "GAMS - A User's Guide." GAMS Development Corporation, Washington, DC, USA.

Waller, T., Shofer, J. L., Ziliaskopoulos, A. K., (2001). Evaluation with Traffic Assignment Under Demand Uncertainty. Transportation Research Record: Journal of the Transportation Research Board, Volume 1771 / 2001, pg. 69-74

Wardrop, J.G., (1952) Some Theoretical Aspects of Road Traffic Research. *Proceedings, Institution of Civil Engineers* II(1), 325-378.

Yin, Y., Ieda, H., (2001). Assessing performance reliability of road networks under nonrecurrent congestion. Transportation Research Record 1771, 148–155.

Zheng Li, David A. Hensher, John M. Rose (2010), Willingness to pay for travel time reliability in passenger transport: A review and some new empirical evidence, *Transportation Research Part E*, Volume 46, Issue 3, Pages 384-40